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A study of fuzzy functions by fuzzy polynomials

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Abstract: In this paper, the problem of best approximation is considered for fuzzy function, by optimization to obtain a fuzzy polynomials. In this work we show this method with using trapezoidal fuzzy numbers.

Key words: approximation; Fuzzy number; Trapezoidal fuzzy; fuzzy polynomial.

INTRODUCTION

The interpolation problem of fuzzy data first was introducted by Zadeh (1965). Lown (1990) gave a fuzzy Lagrange interpolation theorem. Kaleva (1994) represented some properties of Lagrange and cubic splines of odd degree are introduced. Properties of natural splines and complete splines of odd degree are introduced by abbasbandy *et al.*, (1998, 2003).

Interpolation problem is the following : Given $n+1$ different points in *R* with the corresponding fuzzy value in *R*, to find a fuzzy polynomial of degree at most n which consider on these points, with the given fuzzy value (Abbasbandy, 2006)

In this work we first introduce the basic concepts of fuzzy numbers, and a kind of fuzzy polynomial $\tilde{p}_n(x)$ of degree at most n where $x \in R$. Here we are given m distinct in ***R***, and for each points we have a fuzzy value in ***R***. in this work we want to find a fuzzy polynomial of degree at most n. the authors already introduced some methods for solving such problems in (Horngren *et al*., 2005).

In section 3 we introduce best approximation of fuzzy function and a method for computing it with respect to using linear programming. But dose the best approximation of fuzzy function always exist? And is it unique? We will answer these questions in section 4. At last some examples are given in section 5, and conclusion in section 6.

2 Basic concepts:

Let $F(R)$ be the set of all real fuzzy numbers and $T_r(R)$ be the set of all trapezoidal fuzzy numbers, which is

$$\mu_{\tilde{v}}(x) = \begin{cases} \dfrac{x-v_1}{v_2-v_1}, & v_1 \le x \le v_2, \\ 1, & v_2 \le x \le v_3, \\ \dfrac{v_4-x}{v_4-v_3}, & v_3 \le x \le v_4, \\ 0, & otherwise \end{cases}$$

And denoted by $\tilde{v} = (v_1, v_2, v_3, v_4)$. Let $\tilde{u}$ and $\tilde{v}$ be two fuzzy numbers where $\tilde{v} = (v_1, v_2, v_3, v_4)$ and $\tilde{u} = (u_1, u_2, u_3, u_4)$ then $\tilde{u}$ and $\tilde{v}$ satisfying the following results:

1. $x \ge 0 : x\tilde{v} = (xv_1, xv_2, xv_3, xv_4)$
2. $x < 0 : x\tilde{v} = (xv_4, xv_3, xv_2, xv_1)$
3. $(\tilde{v} + \tilde{u}) = (v_1 + u_1, v_2 + u_2, v_3 + u_3, v_4 + u_4)$
4. $(\tilde{v} - \tilde{u}) = (v_1 - u_4, v_2 - u_3, v_3 - u_4, v_4 - u_1)$

Definition 2.1:

For two trapezoidal fuzzy numbers $\tilde{v} = (v_1, v_2, v_3, v_4)$ and $\tilde{u} = (u_1, u_2, u_3, u_4)$ the quantity

$$D(\tilde{v}, \tilde{u}) = ((v_1-u_1)^2 + (v_2-u_2)^2 + (v_3-u_3)^2 + (v_4-u_4)^2)^{\frac{1}{2}} \tag{1}$$

Defines a distance for two trapezoidal fuzzy number $\tilde{v}$ and $\tilde{u}$.

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3 Best approximation of a fuzzy function:

Let $X = \{x_1, x_2, \dots, x_m\}$ be a set of m distinct points of R, and $\tilde{f}(x_i)$, be the value of a fuzzy function $\tilde{f} : R \to T_r F(r)$, at these point are $\left(x_i, \tilde{f}(x_i) = (f_{i1}, f_{i2}, f_{i3}, f_{i4})\right)$, for $i = 1,2, \dots, m$.

Definition 3.1:

For every polynomial $\tilde{p}_n(x) \in \widetilde{\prod}_n$, that $\widetilde{\prod}_n$, is the set of all fuzzy polynomials from degree at most n which

$$\tilde{p}_n(x) = \sum_{j=0}^{n} \tilde{a}_j x^j \tag{2}$$

Where $\tilde{a}_j = (a_{j1}, a_{j2}, a_{j3}, a_{j4})$, and so $a_{j1} \le a_{j2} \le a_{j3} \le a_{j4}$.
It obvious that

$$\tilde{p}_n(x_i) = \sum_{j=0}^{n} (\tilde{a}_j x_i^j) = \sum_{j=0}^{n} (a_{j1}, a_{j2}, a_{j3}, a_{j4})\, x_i^j =$$

$$\Big(\sum_{x_i^j \ge 0} a_{j1} x_i^j + \sum_{x_i^j < 0} a_{j4} x_i^j + \sum_{x_i^j \ge 0} a_{j2} x_i^j + \sum_{x_i^j < 0} a_{j3} x_i^j + \sum_{x_i^j \ge 0} a_{j3} x_i^j + \sum_{x_i^j < 0} a_{j2} x_i^j + \sum_{x_i^j \ge 0} a_{j4} x_i^j + \sum_{x_i^j < 0} a_{j1} x_i^j\Big)$$

To find best of approximation $\tilde{p}_n(x)$, we should minimize $D\left(p_n(x_i), f(x_i)\right)$ (for $i = 1,2, \dots m$).
So we want find $\tilde{p}_n(x)$ with linear programming problem. Therefore we choose θ such that

$$\begin{array}{ll} \min \quad \theta & \\ s.t & \\ D\left(p_n(x_i), f(x_i)\right) \le \theta & i = 1,2, \dots, m \\ a_{j2} - a_{j1} \ge 0 & j = 1,2, \dots, n \\ a_{j3} - a_{j2} \ge 0 & j = 1,2, \dots, n \\ a_{j4} - a_{j3} \ge 0 & j = 1,2, \dots, n \end{array} \tag{3.1}$$

We have that

$$D\left(p_n(x_i), f(x_i)\right) =$$

$$\Big(\big(\sum_{x_i^j \ge 0} a_{j1} x_i^j + \sum_{x_i^j < 0} a_{j4} x_i^j - f_{i1}\big)^2 + \big(\sum_{x_i^j \ge 0} a_{j2} x_i^j + \sum_{x_i^j < 0} a_{j3} x_i^j - f_{i2}\big)^2$$
$$+ \big(\sum_{x_i^j \ge 0} a_{j3} x_i^j + \sum_{x_i^j < 0} a_{j2} x_i^j - f_{i3}\big)^2 + \big(\sum_{x_i^j \ge 0} a_{j4} x_i^j + \sum_{x_i^j < 0} a_{j1} x_i^j - f_{i4}\big)^2\Big)^{\frac{1}{2}} \le \theta$$

It is obvious that

$$\big(\sum_{x_i^j \ge 0} a_{j1} x_i^j + \sum_{x_i^j < 0} a_{j4} x_i^j - f_{i1}\big)^2 \le \theta_1^2 \qquad i = 1,2, \dots, m$$

$$
\begin{aligned}
&(\sum_{x_i^j \geq 0} a_{j2}x_i^j + \sum_{x_i^j < 0} a_{j3}x_i^j - f_{i2})^2 \leq \theta_2^2 \qquad i = 1,2,\dots,m \\
&(\sum_{x_i^j \geq 0} a_{j3}x_i^j + \sum_{x_i^j < 0} a_{j2}x_i^j - f_{i3})^2 \leq \theta_3^2 \qquad i = 1,2,\dots,m \\
&(\sum_{x_i^j \geq 0} a_{j4}x_i^j + \sum_{x_i^j < 0} a_{j1}x_i^j - f_{i4})^2 \leq \theta_4^2 \qquad i = 1,2,\dots,m
\end{aligned} \tag{3.2}
$$

We have that

$$
\begin{aligned}
&\left|\sum_{x_i^j \geq 0} a_{j1}x_i^j + \sum_{x_i^j < 0} a_{j4}x_i^j - f_{i1}\right| \leq \theta_1 \qquad i = 1,2,\dots,m \\
&\left|\sum_{x_i^j \geq 0} a_{j2}x_i^j + \sum_{x_i^j < 0} a_{j3}x_i^j - f_{i2}\right| \leq \theta_2 \qquad i = 1,2,\dots,m \\
&\left|\sum_{x_i^j \geq 0} a_{j3}x_i^j + \sum_{x_i^j < 0} a_{j2}x_i^j - f_{i3}\right| \leq \theta_3 \qquad i = 1,2,\dots,m \\
&\left|\sum_{x_i^j \geq 0} a_{j4}x_i^j + \sum_{x_i^j < 0} a_{j1}x_i^j - f_{i4}\right| \leq \theta_4 \qquad i = 1,2,\dots,m
\end{aligned} \tag{3.3}
$$

Therefore with (3.3), model (3.1) rewrite as follow

$$
\begin{aligned}
&min \quad \theta_1 + \theta_2 + \theta_3 + \theta_4 \\
&s.t \\
&\sum_{x_i^j \geq 0} a_{j1}x_i^j + \sum_{x_i^j < 0} a_{j4}x_i^j - \theta_1 \leq f_{i1} \qquad i = 1,2,\dots,m \\
&-\sum_{x_i^j \geq 0} a_{j1}x_i^j - \sum_{x_i^j < 0} a_{j4}x_i^j - \theta_1 \leq -f_{i1} \qquad i = 1,2,\dots,m \\
&\sum_{x_i^j \geq 0} a_{j2}x_i^j + \sum_{x_i^j < 0} a_{j3}x_i^j - \theta_2 \leq f_{i2} \qquad i = 1,2,\dots,m \\
&-\sum_{x_i^j \geq 0} a_{j2}x_i^j - \sum_{x_i^j < 0} a_{j3}x_i^j - \theta_2 \leq -f_{i2} \qquad i = 1,2,\dots,m \\
&\sum_{x_i^j \geq 0} a_{j3}x_i^j + \sum_{x_i^j < 0} a_{j2}x_i^j - \theta_3 \leq f_{i3} \qquad i = 1,2,\dots,m \\
&-\sum_{x_i^j \geq 0} a_{j3}x_i^j - \sum_{x_i^j < 0} a_{j2}x_i^j - \theta_3 \leq -f_{i3} \qquad i = 1,2,\dots,m
\end{aligned} \tag{3.4}
$$

$$\sum_{x_i^j \geq 0} a_{j4} x_i^j + \sum_{x_i^j < 0} a_{j1} x_i^j - \theta_4 \leq f_{i4} \qquad i = 1,2,\ldots,m$$
$$-\sum_{x_i^j \geq 0} a_{j4} x_i^j - \sum_{x_i^j < 0} a_{j1} x_i^j - \theta_4 \leq -f_{i4} \qquad i = 1,2,\ldots,m$$
$$a_{j2} - a_{j1} \geq 0 \qquad j = 1,2,\ldots,n$$
$$a_{j3} - a_{j2} \geq 0 \qquad j = 1,2,\ldots,n$$
$$a_{j4} - a_{j3} \geq 0 \qquad j = 1,2,\ldots,n$$

Therefore, using (3,4), we find a_{j1}, a_{j2}, a_{j3} and a_{j4} with above linear programming. Then we find the interpolation polynomial $\tilde{p}_n(x_i) = \sum_{j=0}^{n} (\tilde{a}_j x_i^j)$ that $\tilde{a}_j = (a_{j1}, a_{j2}, a_{j3}, a_{j4})$ for $(j = 0,1,\ldots,n)$.

4. Existence of best approximation of a fuzzy function:

Theorem 4.1 Best approximation of fuzzy function exists.

Proof: For the purpose of existence of universal approximation of a fuzzy function, we show that the problem (3,4) has a solution.

Let $\tilde{f}$ be arbitrary fuzzy function whose values on the points of $X = \{x_1, x_2, \ldots, x_m\}$ are $\tilde{f}(x_1), \tilde{f}(x_2), \ldots, \tilde{f}(x_m)$ that $\tilde{f}(x_m) = (f_{i1}(x_i), f_{i2}(x_i), f_{i3}(x_i), f_{i4}(x_i))$ for $i = 1,2,\ldots,m$.
By a simple computation (3.4), we can be shown that we should solve following linear programming problem:

$$\begin{aligned}
&\min z = \theta_1 + \theta_2 + \theta_3 + \theta_4 \\
&s.t \\
&AX_1 \geq c_1 \\
&AX_2 \geq c_2 \\
&AX_3 \geq c_3 \\
&AX_4 \geq c_4 \\
&a_{j2} - a_{j1} \geq 0 \qquad j = 1,2,\ldots,n \\
&a_{j3} - a_{j2} \geq 0 \qquad j = 1,2,\ldots,n \\
&a_{j4} - a_{j3} \geq 0 \qquad j = 1,2,\ldots,n
\end{aligned} \tag{3.5}$$

It is clear that $X_1 = (\theta_1, 0, \ldots, 0), X_2 = (\theta_2, 0, \ldots, 0), X_3 = (\theta_3, 0, \ldots, 0), X_4 = (\theta_4, 0, \ldots, 0)$ is a feasible solution of (3.4), such that

$$\theta_1 = \sum_{f_{i1}(x_i) \geq 0} f_{i1}(x_i) - \sum_{f_{i1}(x_i) < 0} f_{i1}(x_i) \qquad i = 1,2,\ldots,m$$
$$\theta_2 = \sum_{f_{i2}(x_i) \geq 0} f_{i2}(x_i) - \sum_{f_{i2}(x_i) < 0} f_{i2}(x_i) \qquad i = 1,2,\ldots,m$$
$$\theta_3 = \sum_{f_{i3}(x_i) \geq 0} f_{i3}(x_i) - \sum_{f_{i3}(x_i) < 0} f_{i3}(x_i) \qquad i = 1,2,\ldots,m$$
$$\theta_4 = \sum_{f_{i4}(x_i) \geq 0} f_{i4}(x_i) - \sum_{f_{i4}(x_i) < 0} f_{i4}(x_i) \qquad i = 1,2,\ldots,m$$

Because

$$\min z = \theta_1 + \theta_2 + \theta_3 + \theta_4$$

$s.t$

$$\theta_1\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\geq\begin{pmatrix}f_{1_1}\\\vdots\\f_{m_1}\\-f_{1_1}\\\vdots\\-f_{m_1}\end{pmatrix}=c_1$$

$$\theta_2\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\geq\begin{pmatrix}f_{1_2}\\\vdots\\f_{m_2}\\-f_{1_2}\\\vdots\\-f_{m_2}\end{pmatrix}=c_2$$

$$\theta_3\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\geq\begin{pmatrix}f_{1_3}\\\vdots\\f_{m_3}\\-f_{1_3}\\\vdots\\-f_{m_3}\end{pmatrix}=c_3$$

$$\theta_4\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\geq\begin{pmatrix}f_{1_4}\\\vdots\\f_{m_4}\\-f_{1_4}\\\vdots\\-f_{m_4}\end{pmatrix}=c_4$$

We know $z \geq 0$ (by definition of $\theta_1, \theta_2, \theta_3, \theta_4$). Thus the set of all z for feasible points, is bounded from below, and (4.1) has a solution. It means that the universal approximation of fuzzy function exist.

5 Numerical examples:

Example 5.1: *Here we consider* $m = 2$ *and* $n = 1$

x	1	2
f(x)	(0,1,2)	(1,2,3)

$$\tilde{p}_1(x) = \sum_{j=0}^{1} \tilde{a}_j x^j = \tilde{a}_1 x + \tilde{a}_0 = (0.3333,1,1)x + (0,0,1)$$

Example 5.2: *Here we consider* $m = 3$ *and* $n = 2$

x	0	1	2
f(x)	(0,1,2,3)	(-14,2,5,14)	(-46,5,12,46)

$$\tilde{p}_2(x) = \sum_{j=0}^{2} \tilde{a}_j x^j = \tilde{a}_2 x^2 + \tilde{a}_1 x + \tilde{a}_0 = (0,1,2,10.3333)x^2 + (0,0,1,1)x + (0,1,2,2.8333)$$

Example 5.3 *Here we consider* $m = 5$ *and* $n = 2$

x	0	0.5	1	1.5	2
f(x)	(0,1,2,3)	(-1,1,2,4)	(1,2,4,5)	(1,3,5,7)	(0,1,3,4)

$$\tilde{p}_2(x) = \sum_{j=0}^{2} \tilde{a}_j x^j = \tilde{a}_2 x^2 + \tilde{a}_1 x + \tilde{a}_0$$
$$= (0,0,0,0.25)x^2 + (0,0,0.6667,0)x + (0,2,2.8333,4.7188)$$

Conclusion:

Finding of polynomial approximation of a function has benefit properties. Some researchers found interpolating polynomial and some others solved this problem. But we introduced a new method which differs from both. In this paper we propose a method to find a fuzzy polynomial as a best approximation of a fuzzy function on a discrete set of points.

We can find a polynomial of degree of n that n can be arbitrary. Also we can approximate a trapezoidal fuzzy function $\tilde{f}$. However it is not possible to approximate it by any other methods. The best approximation of fuzzy function is exists. It means that we can approximate any tabular function when all $\tilde{f}(x_i)$'s are a trapezoidal fuzzy numbers.

DS has been used for computations in this paper.

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