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On Control Problem Described by Infinite System of First-Order Differential Equations

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Abstract: We study **c**ontrol problem described by infinite system of first order differential equations in Hilbert Space. Control parameter is subjected to integral constraint. Our goal is to transfer the state of the system from a given initial position to the origin for a finite time. In this paper, we obtained necessary and sufficient condition for which the goal to be achieved. Control function is constructed in an explicit form. Furthermore, equation for calculating optimal time for the transfer process is obtained.

Key words: Infinite system, Control, Integral constraint, Hilbert space.

INTRODUCTION

The importance of control problems motivated the development of many books on fundamental results (see, for example, Avdonin and Ivanov, 1989: Butkovskiy, 1975: Egorov, 2004: Kirk, 1998: Pinch, 1993: and Pontryagin *et al*., 1969).

Control problems described by partial differential are of significant importance in solving real life problems and in other research areas such as engineering and economics. This forms a motivation for the extensive study of such type of problems by many researchers using different approaches (e.g. Avdonin and Ivanov, 1989: Butkovskiy, 1975: Chernous'ko, 1992: Ibragimov, 2003: Ilin, 2001: Osipov, 1977: Tukhtasinov and Mamatov, 2009: and Satimov and Tukhtasinov, 2006). Among the approaches found in literature is the use of decomposition method to reduce the problem to the one described by an infinite system of differential equations. (see, for example, Butkovskiy, 1975: Chernous'ko, 1992: Ibragimov, 2003: Tukhtasinov and Mamatov, 2009: and Satimov and Tukhtasinov, 2006).

In Chernous'ko, 1992, a control system described by the following partial differential equation

$$u_t = Au + w, \tag{1}$$

was reduced to the one described by the system of ordinary differential equations

$$\dot{z}_k(t) + \mu z_k(t) = w_k(t),\ k = 1, 2, ..., \tag{2}$$

and investigated, where in (1) $u = u(t, x)$ is a scalar function of $x \in R^n$ and time t; w is a control parameter;

$$Au = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right)$$

is a linear differential operator whose coefficients do not depend on t. In (2), $w_k, k = 1, 2, ...,$ are control parameters and constants $\mu_k, k = 1, 2, ...,$ satisfy the condition

$$0 \le \mu_1 \le \mu_2 \le \to \infty.$$

Furthermore, in Tukhtasinov and Mamatov, 2009; and Satimov and Tukhtasinov, 2006 control problems described by (1) consisting of two control system with conflicting goals under different forms of restrictions were investigated. The decomposition methods were also employed.

This approach hints the significant relationship between control problems described by partial differential equations and those described by infinite system of differential equations. Therefore, the latter can be studied in a separate theoretical framework. For instance, in Ibragimov and Hasim, 2010, pursuit and evasion differential

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game described by (2) was studied in a framework independent of that described by partial differential equations.

In this paper, we investigate control problem described by the system (2) in the case of negative coefficients μ_k, in a framework different from the one described by partial differential equations.

Statement of the problem:

Let $\lambda_1, \lambda_2, \ldots,$ be a bounded sequence of negative numbers and r be a real number. We introduce the space

$$l_r^2 = \left\{ \alpha = (\alpha_1, \alpha_2, \ldots) : \sum_{k=1}^{\infty} |\lambda_k|^r \alpha_k^2 < \infty \right\},$$

with inner product and norm

$$\langle \alpha, \beta \rangle_r = \sum_{k=1}^{\infty} |\lambda_k|^r \alpha_k \beta_k, \quad \alpha, \beta \in l_r^2, \quad \|\alpha\|_r = \left(\sum_{k=1}^{\infty} |\lambda_k|^r \alpha_k^2 \right)^{1/2}.$$

Let

$$L_2(0,T; l_r^2) = \left\{ w(t) = (w_1, w_2, \ldots) : \sum_{k=1}^{\infty} |\lambda_k|^r \int_0^T w_k^2(t)dt < \infty, \; w_k(\cdot) \in L_2(0, T) \right\},$$

$$\|w(\cdot)\|_{L_2(0,T;l_r^2)} = \left(\sum_{k=1}^{\infty} |\lambda_k|^r \int_0^T w_k^2(t)dt \right)^{1/2},$$

where $T, \; T > 0,$ is a given number.

We examine a control problem described by the following system of differential equations

$$\dot{z}_k(t) + \lambda_k z(t) = w_k(t), \; z_k(0) = z_k^0, \; , \; k = 1, 2, \ldots, \tag{3}$$

where $z_k, w_k, \in R^1, \; k = 1, 2, \ldots, \; z_0 = (z_1^0, \; z_2^0, \ldots) \in l_{r+1}^2, \; w_1, w_2, \ldots,$ are control parameters.

Definition 1:

A function $w(\cdot)$, $w : [0, T] \to l_r^2$, with measurable coordinates subject to

$$\sum_{k=1}^{\infty} |\lambda_k|^r \int_0^T w_k^2(\tau)d\tau \le \rho^2,$$

where ρ is a given positive number, is referred to as the admissible control.

We denote the set of all admissible controls by $S(\rho)$.

Definition 2:

A function $z(t) = (z_1(t), z_2(t), \ldots), \; 0 \le t \le T,$ is called the solution of the system (3) if each coordinates $z_k(t)$ of that

1) *is continuously differentiable on $(0,T)$, and satisfies the initial conditions $z_k(0) = z_k^0$;*
2) *has the first derivative $\dot{z}_k(t)$ almost everywhere on $(0,T)$ and satisfies the equation*

$$\dot{z}_k(t) + \lambda_k z(t) = w_k(t), \quad k = 1, 2, \ldots, \textit{ almost everywhere on } (0,T).$$

Definition 3:

The number $\theta > 0$ is an optimal control time if

1) *there exists an admissible control to ensure that* $z(\theta)=0$;
2) *there exists no admissible control to ensure* $z(\tau)=0$ *for any* $\tau\in(0,\theta)$.

The problem is to find a control $w(\cdot)=(w_1(\cdot),w_2(\cdot),...)$ such that the state of the system (3) can be transferred to the origin for a finite time.

Main result:

It is not difficult to verify that the kth equation in (3) has a unique solution

$$z_k(t) = z_k^0 e^{\alpha_k t} + \int_0^t w_k(\tau) e^{\alpha_k(t-\tau)} d\tau. \tag{4}$$

where $0<\alpha_k=-\lambda_k$.

Let $C(0,T;\, l_r^2)$ be the space of continuous functions $z(t)=\left(z_1(t),\ z_2(t),...\right)$, with values in l_r^2. The following assertion can be proved similar to (Avdonin and Ivanov, 1989).

Assertion 1:

If $\lambda_k<0,\ k=1,2,...,$ *is a bounded below sequence, then the function* $z(t)=\left(z_1(t),\ z_2(t),...\right)$ *defined by (4) belongs to the space* $C(0,T;\, l_{r+1}^2)$.

Let $$F(t)=\sum_{k=1}^{\infty}\alpha_k^r\psi_k(t)z_{k0}^2,\ \ \psi_k(t)=\frac{2\alpha_k}{1-e^{-2\alpha_k t}}. \tag{5}$$

Lemma 1:

Let $z_0\in l_{r+1}^2$. *The series in (5) is convergent at any fixed* $t>0$ *if and only if* $z_0\in l_r^2$.

Proof. Suppose that $\sum_{k=1}^{\infty}\alpha_k^r\psi_k(t)z_{k0}^2$ is convergent. Let $\phi(\alpha)=\dfrac{2\alpha}{1-e^{-2\alpha t}},\ \alpha>0$. The inequality

$$\phi(\alpha)\ge\frac{1}{t},\ t>0,$$

is true, since $\phi(\alpha)$ is increasing and approaches $\frac{1}{t}$ as α approaches zero. Therefore, we have

$$\sum_{k=1}^{\infty}\alpha_k^r\psi_k(t)z_{k0}^2=\sum_{k=1}^{\infty}\alpha_k^r z_{k0}^2\frac{2\alpha_k}{1-e^{-2\alpha_k t}}\ge\frac{1}{t}\sum_{k=1}^{\infty}\alpha_k^r z_{k0}^2.$$

Hence,

$$\sum_{k=1}^{\infty}\alpha_k^r z_{k0}^2<\infty,$$

and so $z_0\in l_r^2$.

Conversely, let $z_0\in l_r^2$. We have

$$F(t)=\sum_{k=1}^{\infty}\alpha_k^r z_{k0}^2\frac{2\alpha_k}{1-e^{-2\alpha_k t}} \tag{6}$$

$$=\sum_{\alpha_k<1}\alpha_k^r z_{k0}^2\frac{2\alpha_k}{1-e^{-2\alpha_k t}}+\sum_{\alpha_k\ge1}\alpha_k^r z_{k0}^2\frac{2\alpha_k}{1-e^{-2\alpha_k t}},$$

therefore, the series (6) is convergent if the series

$$\sum_{\alpha_k<1} \alpha_k^r z_{k0}^2 \frac{2\alpha_k}{1-e^{-2\alpha_k t}}, \tag{7}$$

And

$$\sum_{\alpha_k\geq 1} \alpha_k^r z_{k0}^2 \frac{2\alpha_k}{1-e^{-2\alpha_k t}} \tag{8}$$

are convergent. As the function $\phi(\alpha)$ is increasing, then replacing α_k by 1 (since $\alpha_k < 1$) in (7) we obtain

$$\sum_{\alpha_k<1} \alpha_k^r z_{k0}^2 \frac{2\alpha_k}{1-e^{-2\alpha_k t}} \leq \frac{2}{1-e^{-2t}} \sum_{\alpha_k<1} \alpha_k^r z_{k0}^2 \quad .$$

Therefore, the series (7) is convergent, since $z_0 \in l_r^2$.
Convergence of the series (8) follows from the relations

$$\sum_{\alpha_k\geq 1} \alpha_k^r z_{k0}^2 \frac{2\alpha_k}{1-e^{-2\alpha_k t}} = \sum_{\alpha_k\geq 1} 2\alpha_k^{r+1} z_{k0}^2 \frac{1}{1-e^{-2\alpha_k t}} \leq \frac{2}{1-e^{-2t}} \sum_{\alpha_k\geq 1} \alpha_k^{r+1} z_{k0}^2 < \infty$$

This is because

$$\varphi(\alpha) = \frac{1}{1-e^{-2\alpha t}}, \ \alpha \geq 1, \ t > 0$$

is decreasing function and $z_0 \in l_{r+1}^2$.

Convergence of (7) and (8) imply the convergence of the series (6). The proof of the lemma is complete.

Moreover, if $z_0 \in l_r^2 \cap l_{r+1}^2$, then the function $F(t)$ has the following properties:

(i) $F(t)$ is a decreasing function of t ,
(ii) $F(t)$ approaches $+\infty$ as $t \to 0$,
(iii) $F(t)$ approaches $\sum_{k=1}^{\infty} 2\alpha_k^{r+1} z_{k0}^2$, as $t \to +\infty$
(iv) $F(t)$ is continuous.

Indeed, property (i) follows from the fact that each term of the series is decreasing function of t . Since each term of $F(t)$ approaches $+\infty$ as $t \to 0$, therefore property (ii) is true. We now prove property (iii).
If $z_0 \in l_r^2 \cap l_{r+1}^2$, then $F(t)$ is convergent for any $t > 0$. We fix t . As $F(t)$ is convergent, then for any $\varepsilon > 0$ there exists a number N such that

$$\sum_{k=N+1}^{\infty} \alpha_k^r \psi_k(t) z_{k0}^2 < \frac{\varepsilon}{2}, \tag{9}$$

And

$$\sum_{k=N+1}^{\infty} 2\alpha_k^{r+1} z_{k0}^2 < \frac{\varepsilon}{2} \ . \tag{10}$$

Inequality (9) keeps to hold if we increase t since $\psi_k(t)$, $k = 1,2,...,$ are decreasing functions. There exists T_0 such that

$$\left| \sum_{k=1}^{N} \alpha_k^r \psi_k(t) z_{k0}^2 - \sum_{k=1}^{N} 2\alpha_k^{r+1} z_{k0}^2 \right| < \frac{\varepsilon}{2}$$

whenever $t > T_0$, since $\psi_k(t)$ approaches $2\alpha_k$ as t approaches ∞ and the sum consists of a finite number of summands.

We now have

$$\left|\sum_{k=1}^{\infty} \alpha_k^r \psi_k(t) z_{k0}^2 - \sum_{k=1}^{\infty} 2\alpha_k^{r+1} z_{k0}^2\right|$$

$$\leq \left|\sum_{k=1}^{N} \alpha_k^r \psi_k(t) z_{k0}^2 - \sum_{k=1}^{N} 2\alpha_k^{r+1} z_{k0}^2\right| + \left|\sum_{k=N+1}^{\infty} \alpha_k^r \psi_k(t) z_{k0}^2 - \sum_{k=N+1}^{\infty} 2\alpha_k^{r+1} z_{k0}^2\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Hence, property (iii) is proved.

Finally, we prove property (iv). Continuity of $F(t)$ follows from the fact that it is convergent and $\psi_k(t) = \frac{2}{1-e^{-2\alpha_k t}}$, $k = 1, 2, ...,$ $t > 0,$ are continuous.

We now have from the property (i) and (iii) that

$$F(t) > \sum_{k=1}^{\infty} 2\alpha_k^{r+1} z_{k0}^2, \ \forall \ t > 0. \qquad (11)$$

Now, consider the equation

$$F(t) = \sigma^2 \qquad (12)$$

If equation (12) has a solution $t = \theta$, then according to (11)

$$\sigma^2 > \sum_{k=1}^{\infty} 2\alpha_k^{r+1} z_{k0}^2 . \qquad (13)$$

Conversely, let (13) be true. Since $F(t)$ approaches infinity as t approaches zero; $F(t)$ approaches $2\sum_{k=1}^{\infty} \alpha_k^{r+1} z_{k0}^2$ as t approaches infinity and $F(t)$ is continuous and decreasing, there exists a unique $t = \theta$ such that $F(\theta) = \sigma^2$. Therefore, the following assertion is true.

Assertion 2:

The equation defined by (12) has a root if and only if (13) is true and this root is unique.

Theorem 1:

There exists an admissible control to steer the state of the system (3) into the origin if and only if (13) holds.
In the proof of this theorem we use the following lemma. Let

$$\Omega = \left\{ \{ w(\cdot) = (w_1(\cdot), w_2(\cdot), \ldots) \mid \int_0^T w_k(s) e^{-\alpha_k s} ds = \xi_k, \ w(\cdot) \in S(\rho), \ 0 \leq t \leq T \right\}.$$

Lemma 2:

Among all the controls $w(\cdot) \in \Omega$*, the control* $w(\cdot)$ *such that*

$$w_k(t) = \psi_k(T) e^{-\alpha_k t} \xi_k, \ k = 1, 2, ...,$$

minimizes the functional

$$Q(T) = \sum_{k=1}^{\infty} \alpha_k^r \int_0^T w_k^2(s) ds .$$

The proof of this lemma is similar to the proof of the lemma contained in (Ibragimov, 2003) with $-\alpha_k = \lambda_k$.

Proof of Theorem 1:
Assume on the contrary, that (13) is not true

$$\sigma^2 \le \sum_{k=1}^{\infty} 2\alpha_k^{r+1} z_{k0}^2, \tag{14}$$

and that there exists an admissible control $w(\cdot)$ such that $z(\tau) = 0$ for some time τ. Then (using (4))

$$\int_0^{\tau} w_k(s) e^{-\alpha_k s} ds = -z_{k0}, \ k = 1, 2, \dots . \tag{15}$$

By the lemma, among all controls $w(t)$, $0 \le t \le \tau$, satisfying (15), the control $w(t)$:

$$w_k(t) = -z_{k0} \psi_k(\tau) e^{-\alpha_k t}, \ 0 \le t \le \tau,$$

minimizes the functional $Q(\tau)$ i.e.,

$$Q(\tau) \ge \sum_{k=1}^{\infty} \alpha_k^r \int_0^{\tau} z_{k0}^2 \psi_k^2(\tau) e^{-2\alpha_k t} ds = \sum_{k=1}^{\infty} \alpha_k^r z_{k0}^2 \psi_k(\tau) = F(\tau)$$

Hence,

$$F(\tau) \le Q(\tau). \tag{16}$$

Using (11), (14) and (16) we obtained

$$\sigma^2 \le \sum_{k=1}^{\infty} 2\alpha_k^{r+1} z_{k0}^2 < F(\tau) \le Q(\tau),$$

meaning that $w(t) = (w_1(t), w_2(t), \dots)$, $0 \le t \le \tau$ is not admissible. This is a contradiction.

Conversely, suppose that (13) holds. According to Assertion 2, equation (12) has a unique root $t = \theta$.

1^0. *Construction of the control:*

$$w_k(t) = \begin{cases} \frac{2\alpha_k z_{k0} e^{-\alpha_k t}}{e^{-2\alpha_k \theta} - 1}, & 0 \le t \le \theta, \\ 0, & t > \theta. \end{cases} \tag{17}$$

Admissibility of the constructed control follows from the relations

$$\sum_{k=1}^{\infty} \alpha_k^r \int_0^{\theta} w_k^2(\tau) d\tau = \sum_{k=1}^{\infty} \alpha_k^r \int_0^{\theta} \frac{4\alpha_k^2 z_{k0}^2 e^{-2\alpha_k \tau}}{(e^{-2\alpha_k \theta} - 1)^2} d\tau$$

$$= \sum_{k=1}^{\infty} \alpha_k^r \frac{4\alpha_k^2 z_{k0}^2}{(e^{-2\alpha_k \theta} - 1)^2} \int_0^{\theta} e^{-2\alpha_k \tau} d\tau$$

$$= \sum_{k=1}^{\infty} \alpha_k^r \frac{-2\alpha_k z_{k0}^2}{e^{-2\alpha_k \theta} - 1} = \sigma^2,$$

(here we use (17) and definition of θ in (12)).

2^0. *Steering the system to the origin:* Using (4) and control (17), we have

$$z_k(\theta) = e^{\alpha_k \theta} \left(z_{k0} + \int_0^{\theta} \frac{2 z_{k0} \alpha_k e^{-2\alpha_k \tau}}{(e^{-2\alpha_k \theta} - 1)} d\tau \right)$$

$$= e^{\alpha_k \theta} (z_{k0} - z_{k0}) = 0$$

This completes the proof of the theorem.

Theorem 2:

The number θ, the root of (12) is the optimal time to transfer the initial position of the system (3) into the origin.

Proof: Suppose, on the contrary, that there exists $\tau \in [0, \theta)$ such that $z(\tau) = 0$. Consequently, we have inequality (16) holding (*see Proof of Theorem 1*).
Now considering the fact that the control $w(\cdot)$ satisfies the integral constraint; definition of θ and that the function $F(t)$ is decreasing, we have

$$Q(\tau) \le \sigma^2 = F(\theta) < F(\tau).$$

Therefore,

$$Q(\tau) < F(\tau),$$

and this contradicts (16). Therefore, any initial position $z_0 \neq 0$ of the system (3) cannot be transferred into the origin for the time $t \in [0, \theta)$. Hence, the number θ is optimal transfer time. This completes the proof of the theorem.

Conclusion:

A control problem described by system of first order differential equations in Hilbert space has been studied. The coefficients of the system are considered to be bounded sequence of negative numbers. Control function satisfies integral constraint.

Necessary and sufficient condition for which the control process is possible has been obtained. This includes construction of a control function that brings the system into the origin for a finite time.

Moreover, this necessary and sufficient condition depends on the series $F(t)$. As a result of that, we studied this series and consequently obtained necessary and sufficient condition for the convergence of the series $F(t)$.

Finally, we succeeded in obtaining optimal time for transferring the initial position of the system (3) into the origin. We have given an equation to find this time.

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