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The Variational Iteration Method for an Equation with Integral Conditions Arising in Thermoelasticity

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Abstract: In this paper, using the variational iteration method we solve a model parabolic mixed problem with purely boundary integral conditions arising in the context of thermoelasticity. Finally, illustrative examples are given to show the efficiency of the method.

Key word: Variational iteration method, Integral conditions, Lagrange multiplier, Thermoelasticity

INTRODUCTION

The variational iteration method see He (1997, 1998), which is a modified general Lagrange multiplier method see He et al. (2007), has been shown to solve effectively, easily and accurately, a large call of nonlinear problems with approximations which converge quickly to exact solutions. It was successfully applied to ordinary and partial differential equations see Rafei et al. (2007) and He (1998), recently to delay differential equations see Saadatmandi et al. (2009) and Yu (2008), and other fields see Yu (2008), Abdou et al. (2005), Wang et al. (2007), Bo et al. (2007), Dehghan et al. (2007, 2008), Abbasbandy (2007, 2008) and Abbasbandy et al. (2008).

In this paper, we will be dealing with the following equation:

$$\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}=f(x,t), \qquad (x,t)\in(0,1)\times(0,T], \tag{1}$$

with the initial conditions

$$v(x,0)=V_0(x), \qquad 0\le x\le 1, \tag{2}$$

and the integral boundary conditions

$$\int_0^1 v(x,t)\,dx=E(t), \quad 0\le t\le T, \tag{3}$$

$$\int_0^1 x\,v(x,t)\,dx=G(t), \quad 0\le t\le T, \tag{4}$$

where f,r,E and G are sufficiently regular given functions of the indicated variables and T is a positive constant. This mathematical model studied in Bozuziani (2002), describes the quasi static flexure of a thermoelastic rod, where conditions (3) and (4) represent, respectively, the average and weighted average of the entropy v.

Although an increasing attention has been recently given to evolution problems which involve nonlocal boundary conditions see Beilin (2001), Cannon et al. (1990) and Dehghan et al. (2003, 2006, 2007, 2009), only few works have been consecrated to mixed parabolic problems with purely integral boundary conditions over the spatial domain Bouziani (1996, 2002).

The plan of the paper is as follows. In Section 2, we transform problem $(1)-(4)$ to an equivalent one with homogeneous integral conditions, namely, problem $(7)-(10)$. In Section 3, we give a brief description of the variational iteration method. In Section 4, we apply the VIM for solving the problem $(7)-(10)$. In Section 5, numerical examples are simulated to demonstrate the high performance of proposed method. Finally, some conclusions are summarized in the last section.

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2. Reformulation of the Problem:

For the sake of simplicity, we transform problem $(1)-(4)$ with inhomogeneous conditions (3) and (4) to an equivalent one with homogenous conditions. To do so, we use the following transformation

$$u(x,t)=v(x,t)-R(x,t), \qquad (x,t)\in(0,1)\times[0,T], \tag{5}$$

Where

$$R(x,t)=6\left(2G(t)-E(t)\right)x-2\left(3G(t)-2E(t)\right). \tag{6}$$

Then, the function u is seen to be the solution of the following problem

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=F(x,t), \qquad (x,t)\in(0,1)\times(0,T], \tag{7}$$

with the initial conditions

$$u(x,0)=U_0(x), \qquad 0\le x\le 1, \tag{8}$$

and the integral boundary conditions

$$\int_0^1 u(x,t)\,dx=0, \quad 0\le t\le T, \tag{9}$$

$$\int_0^1 x\,u(x,t)\,dx=0, \quad 0\le t\le T, \tag{10}$$

Where

$$F(x,t)=f(x,t)-\frac{\partial R(x,t)}{\partial t}, \tag{11}$$

$$U_0(x)=V_0(x)-R(x,0). \tag{12}$$

Hence, instead of looking for the function v, we search for the function u. The solution of the problem $(1)-(4)$ will be simply given by the formula $v(x,t)=u(x,t)+R(x,t)$.

3. A Brief Description of the Variational Iteration Method:

To illustrate the basic concepts of the VIM, we consider the following differential equation:

$$L\left[u(x)\right]+N\left[u(x)\right]=g(x), \tag{13}$$

Where L is a linear operator, N is nonlinear operator and $g(x)$ is a given continuous function.

The basic character of the method is to construct a correction functional as follows:

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(s)\left[L\left(u_n(t)\right)+N\left(\tilde{u}_n(t)\right)-g(t)\right]dt, \tag{14}$$

Where λ is a general Lagrange multiplier which can be identified via variational theory. u_n is the nth approximate solution of the VIM, and $\tilde{u}_n$ denotes a restricted variation, i.e. $\delta\tilde{u}_n=0$.

4. Application of the Variational Iteration Method:

According to the variational iteration method, to solve the problem $(7)-(10)$, we can construct the following correction functional:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s)\left(\frac{\partial u_n(x,s)}{\partial s} - \frac{\partial^2 \tilde{u}_n(x,s)}{\partial x^2} - F(x,s)\right)ds, \tag{15}$$

where u_n is n the iteration of the VIM for u.

Calculating variation with respect to u_n, and noting that $\delta \tilde{u}_n = 0$ we obtain:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(s)\left(\frac{\partial u_n(x,s)}{\partial s} - \frac{\partial^2 \tilde{u}_n(x,s)}{\partial x^2} - F(x,s)\right)ds, \tag{16}$$

by parts we have

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \lambda(s)\,\delta u_n(x,s)\big|_{s=t} - \delta\int_0^t \lambda'(s)\, u_n(x,s)ds. \tag{17}$$

For arbitrary δu_n, the following stationary conditions are obtained:

$$\begin{cases} 1+\lambda(t) = 0, \\ \lambda'(s) = 0, \end{cases} \tag{18}$$

Therefore the Lagrange multiplier can be obtained as follows:

$$\lambda(s) = -1 \tag{19}$$

So, the following iteration formula can be obtained as:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,s)}{\partial s} - \frac{\partial^2 u_n(x,s)}{\partial x^2} - F(x,s)\right)ds, \tag{20}$$

and we always begin with $u_0(x,t) = U_0(x)$.

Therefore by the iteration formula (20), we can obtain the numerical solution of the problem $(7)-(10)$.

5. Numerical Examples:

In this section, some numerical examples are simulated to demonstrate the high performance of proposed method.

Example 1:

We consider the following equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0, \qquad (x,t) \in (0,1)\times(0,0.5],$$

with the initial conditions

$$v(x,0) = e^x, \quad 0 \le x \le 1,$$

and the integral boundary conditions

$$\int_0^1 v(x,t)\,dx = e^t(e-1), \quad 0 \le t \le 0.5,$$

$$\int_0^l x\,v(x,t)\,dx = e^t, \qquad 0 \le t \le 0.5.$$

This problem has the exact solution $v(x,t) = e^{x+t}$. From $(6),(11)$ and (12) we have

$$R(x,t) = e^t(12x-6) + e^t(e-1)(-6x+4),$$
$$F(x,t) = -e^t(12x-6) - e^t(e-1)(-6x+4),$$
$$U_0(x) = e^x - \big(12x - 6 + (e-1)(-6x+4)\big).$$

Therefore, we can apply the proposed method in the previous section for solving the problem $(7)-(10)$. Then from (20), we have

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,s)}{\partial s} - \frac{\partial^2 u_n(x,s)}{\partial x^2} - \Big(-e^s(12x-6) - e^s(e-1)(-6x+4)\Big) \right) ds,$$

We use $u_0(x,t) = U_0(x)$ as the initial guess. Then, we use above formula. We can obtain the other iterations as follows

$$v_1(x,t) = e^x(1+t),$$
$$v_2(x,t) = e^x\left(1 + t + \frac{t^2}{2!}\right),$$
$$v_3(x,t) = e^x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right),$$
$$v_4(x,t) = e^x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\right),$$
$$\vdots$$

that gives the exact solution by $v(x,t) = \lim_{n\to\infty} v_n(x,t) = e^{x+t}$, where $v_n(x,t) = u_n(x,t) + R(x,t)$.
We can see the error behavior for 4th, 6th and 8th iterations of VIM in Figs. 1-3.

Example 2:
In this example, we consider the following problem

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \sin(x)\big(\cos(t) - \sin(t)\big), \qquad (x,t) \in (0,1)\times(0,1],$$

with the initial conditions

$$v(x,0) = \sin(x), \quad 0 \le x \le 1,$$

and the integral boundary conditions

$$\int_0^1 v(x,t)\,dx = -\cos(t)\left(\cos(1)-1\right), \quad 0 \le t \le 1,$$

$$\int_0^1 x\,v(x,t)\,dx = \cos(t)\left(-\cos(1)+\sin(1)\right), \quad 0 \le t \le 1,$$

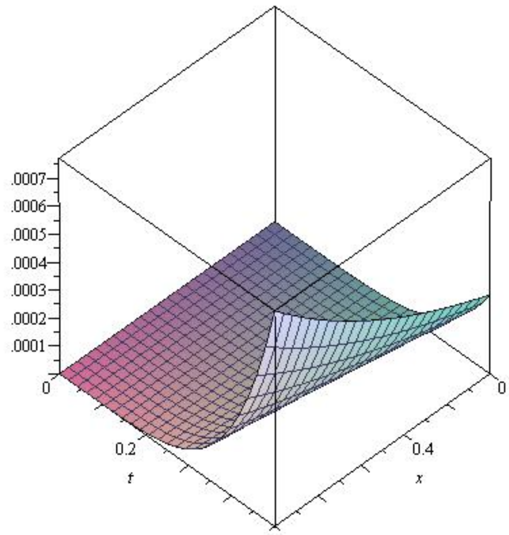

Fig. 1: Plot of the error function $|v(x,t)-v_4(x,t)|$ for Example 1.

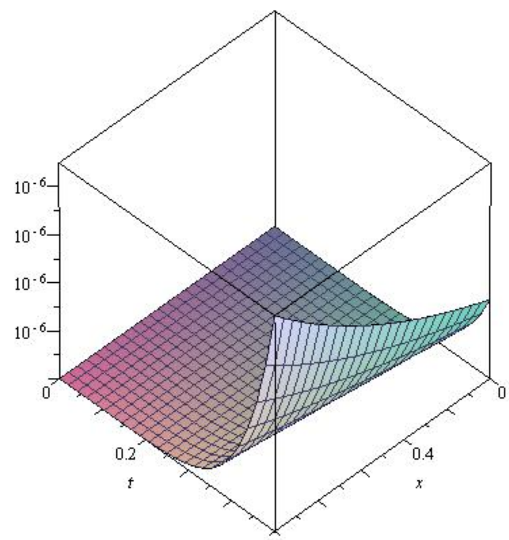

Fig. 2: Plot of the error function $|v(x,t)-v_6(x,t)|$ for Example 1.

The exact solution of this problem is $v(x,t)=\sin(x)\cos(t)$. From $(6), (11)$ and (12) we have

$$R(x,t) = \cos(t)\left(-\cos(1)+\sin(1)\right)(12x-6) - \cos(t)\left(\cos(1)-1\right)(-6x+4),$$
$$F(x,t) = \sin(x)\left(\cos(t)-\sin(t)\right) - \left(-\sin(t)\left(-\cos(1)+\sin(1)\right)(12x-6)+\sin(t)\left(\cos(1)-1\right)(-6x+4)\right),$$
$$U_0(x) = \sin(x) - \left(\left(-\cos(1)+\sin(1)\right)(12x-6) - \left(\cos(1)-1\right)(-6x+4)\right).$$

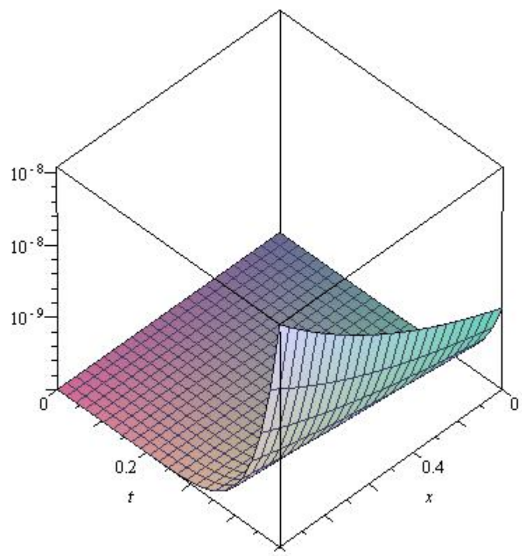

Fig. 3: Plot of the error function $|v(x,t)-v_8(x,t)|$ for Example 1.

Therefore, we can apply the proposed method in the previous section for solving the problem $(7)-(10)$. We use $u_0(x,t)=U_0(x)$ as the initial guess, then from (20), we obtain

$$v_1(x,t)=\sin(x)(-t+\sin(t)+\cos(t)),$$
$$v_2(x,t)=\frac{1}{2}\sin(x)(4\cos(t)+t^2-2),$$
$$v_3(x,t)=-\frac{1}{6}\sin(x)(6\sin(t)-6\cos(t)+t^3-6t),$$
$$v_4(x,t)=\frac{1}{24}\sin(x)(t^4-12t^2+24),$$
$$v_5(x,t)=\frac{1}{120}\sin(x)(120\sin(t)+120\cos(t)-t^5+20t^3-120t),$$
$$v_6(x,t)=\frac{1}{720}\sin(x)(1440\cos(t)+t^6-30t^4+360t^2-720),$$
$$v_7(x,t)=-\frac{1}{5040}\sin(x)(5040\sin(t)-5040\cos(t)+t^7-42t^5+840t^3-5040t),$$
$$\vdots$$

We observe that $v(x,t)=\lim_{n\to\infty}v_n(x,t)=\sin(x)\cos(t)$. Similar to the previous example, it is clear from Figs. 4-6 that the difference between the exact and the numerical solutions is very small for 7^{th}, 8^{th} and 9^{th} iterations of VIM.

Example 3:

In the third example, we consider the following equation

$$\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}=0, \qquad (x,t)\in(0,1)\times(0,0.5],$$

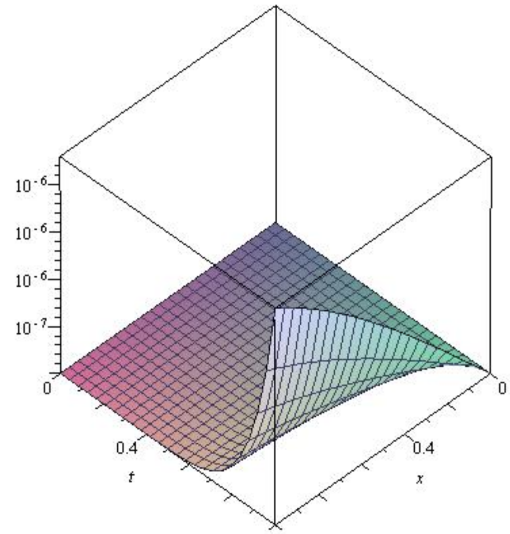

Fig. 4: Plot of the error function $|v(x,t)-v_7(x,t)|$ for Example 2.

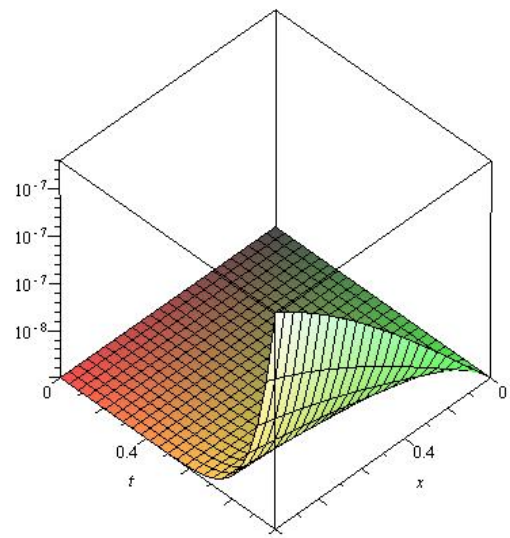

Fig. 5: Plot of the error function $|v(x,t)-v_8(x,t)|$ for Example 2.

with the initial conditions

$$v(x,0)=\cos(x), \qquad 0 \le x \le 1,$$

and the integral boundary conditions

$$\int_0^1 v(x,t)\,dx = e^{-t}\sin(1), \quad 0 \le t \le 0.5,$$

$$\int_0^1 x\,v(x,t)\,dx = e^{-t}\left(\cos(1)+\sin(1)-1\right), \quad 0 \le t \le 0.5,$$

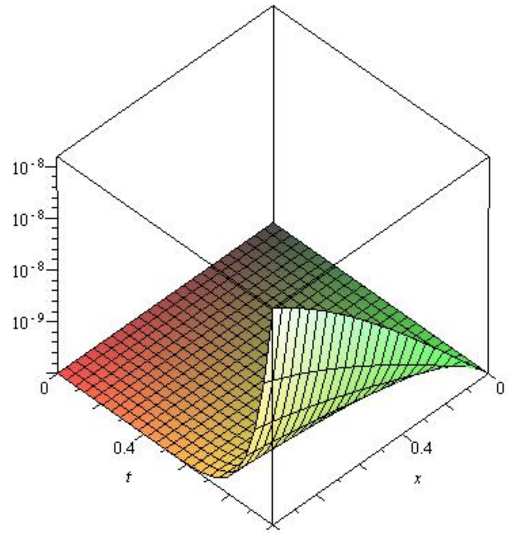

Fig. 6: Plot of the error function $\left|v(x,t)-v_9(x,t)\right|$ for Example 2.

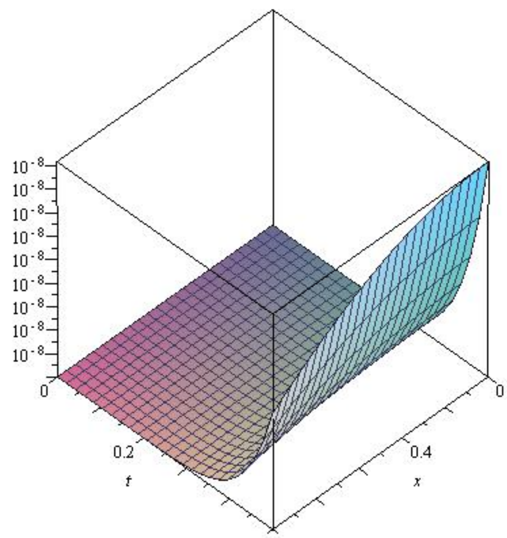

Fig. 7: Plot of the error function $\left|v(x,t)-v_7(x,t)\right|$ for Example 3.

This problem has the exact solution $v(x,t)=e^{-t}\cos(x)$. From $(6),(11)$ and (12) we have

$$R(x,t)=e^{-t}\left(\cos(1)+\sin(1)-1\right)(12x-6)+e^{-t}\sin(1)(-6x+4),$$
$$F(x,t)=e^{-t}\left(\cos(1)+\sin(1)-1\right)(12x-6)+e^{-t}\sin(1)(-6x+4),$$
$$U_0(x)=\cos(x)-\left(\left(\cos(1)+\sin(1)-1\right)(12x-6)+\sin(1)(-6x+4)\right).$$

Now, we can use the proposed method in the previous section for solving the problem $(7)-(10)$. We use $u_0(x,t)=U_0(x)$ as the initial guess, then from (20), we obtain

$$v_1(x,t) = \cos(x)(1-t),$$
$$v_2(x,t) = \cos(x)\left(1-t+\frac{t^2}{2!}\right),$$
$$v_3(x,t) = \cos(x)\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}\right),$$
$$v_4(x,t) = \cos(x)\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}\right),$$
$$v_5(x,t) = \cos(x)\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}\right),$$
$$\vdots$$

It is clear from the above results that $v(x,t) = \lim_{n\to\infty} v_n(x,t) = e^{-t}\cos(x)$.

Similar to the previous Figs. 7-9. show high accuracy of the obtained results for 7th, 8th and 9th iterations of VIM.

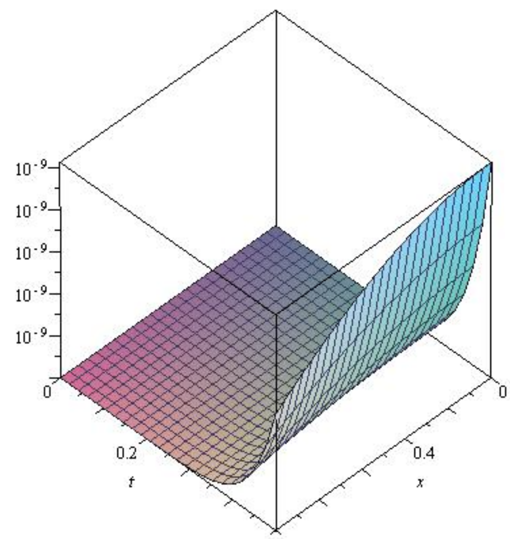

Fig. 8: Plot of the error function $|v(x,t) - v_8(x,t)|$ for Example 3.

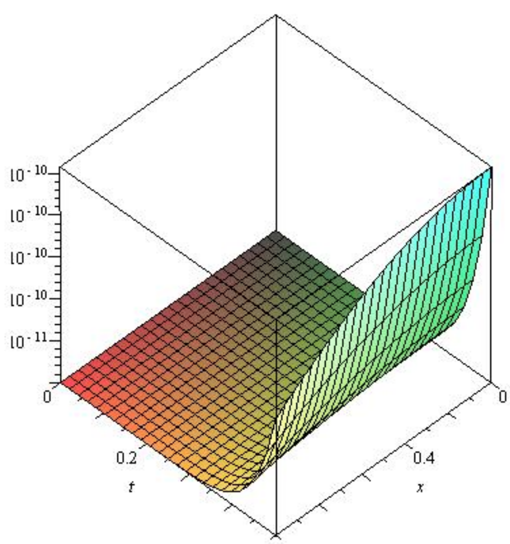

Fig. 9: Plot of the error function $|v(x,t) - v_9(x,t)|$ for Example 3.

Conclusions:

In this work, we investigate a model parabolic mixed problem with purely boundary integral conditions arising in the context of thermoelasticity. For the sake of simplicity, we transform problem with inhomogeneous conditions to an equivalent one with homogenous conditions. Then, We have shown that the variational iteration method can be used successfully for solving this problem. Finally, numerical examples are simulated to demonstrate the high performance of proposed method. All of the computations have done by the Maple software.

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