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A New Characterization of The Janko Group

Seyed Sadegh Salehi Amiri, Alireza Khalili Asboei and Abolfazl Tehranian

Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.

Abstract. Let G be a finite group and $\pi_e(G)$ be the set of elements of order G. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. Set nse(G):={$m_k | k \in \pi_e(G)$}. It is proved that Mathieu groups are uniquely determined by its nse and order. In this paper as the main result, it is proved that if G is a group such that nse(G)=nse(J_1), then $G \cong J_1$.

Key words: Janko group, element order, simple group, Sylow subgroup.

INTRODUCTION

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Denote by $\pi(G)$ the set of primes p such that G contains an element of order p. Also the set of element orders of G is denoted by $\pi_e(G)$. A finite group G is called a simple K_n −group, if G is a simple group with $|\pi(G)| = n$. Set $m_i := m_i(G)=|\{g \in G|$ the order of g is $i\}$ and nse$(G) := \{m_i | i \in \pi_e(G)\}$. In fact, m_i is the number of elements of order i in G and nse(G) is the set of sizes of elements with the same order. In (Shao, C.G., et al., 2008) and (Shao, C.G., Q.H. Jiang. 2010), it is proved that all simple K_4 −groups and Mathieu groups can be uniquely determined by nse(G) and order G. In (Khatami, M. et al., 2009) and (Shen, R., et al., 2010), it is proved that if G is one of the groups A_4, A_5, A_6 and $PSL(2, q)$, for $q \in \{7,8,11,13\}$, then it can be uniquely determined by only nse(G). In this paper, by new method we show that the Janko group J_1 is characterizable by only nse(G). In fact, the main theorem of our paper is as follow:

Main Theorem:
Let G be a group such that nse(G) = nse(J_1), then $G \cong J_1$.

We note that there are finite groups which are not characterizable by nse(G) and $|G|$. In 1987, Thompson gave an example as follows:

Let $G_1 = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$and $G_2 = L_3(4) \rtimes C_2$ be the maximal subgroups of M_{23}. Then nse(G_1) = nse(G_2) and $|G_1|$=$|G_2|$, but $G_1 \ncong G_2$. Throughout this paper, we denote by $\varphi(n)$ the Euler totient function. If G is a finite group, then we denote by P_q a Sylow q −subgroup of G and $n_q := n_q(G)$ is the number of Sylow q −subgroup of G, that is., $n_q(G)$=$|Syl_q(G)|$. All further unexplained notations are standard and refer to (Conway, J. H., et al., 1985), for example.

2. Preliminary Results:

In this section we bring some preliminary lemmas to be used in the proof of main theorem theorem.

Lemma 2.1. (Frobenius, G., 1895) Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m || L_m(G)|$.

Lemma 2.2. (Khosravi, B. and B. Khosravi. 2005) Let S be a sporadic simple group and p be the greatest element of $\pi(S)$. Then S is uniquely determined by $|S|$ and $n_p(S)$.

Lemma 2.3. (Shen, R., et al., 2010) Let G be a group containing more than two elements. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. If $s = sup\{m_k | k \in \pi_e(G)\}$ is finite, then G is finite and $|G| \leq s(s^2 - 1)$.

Let G be a group such that nse(G) = nse(J_1). By Lemma 2.3, we can assume that G is finite. Let m_n be the number of elements of order n. We note that m_n= $k.\varphi(n)$, where k is the number of cyclic subgroups of order n in G. Also we note that if $n > 2$, then $\varphi(n)$ is even. If $n \in \pi_e(G)$, then by Lemma 2.1 and the above notations we have:

$$\begin{cases} \varphi(n) | m_n \\ n | \sum_{d|n} m_d \end{cases} \qquad (*)$$

In the proof of the main theorem, we almost apply $(*)$ and the above comments.

Corresponding Author: Seyed Sadegh Salehi Amiri, Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
E-mail: salehisss@yahoo.com

3. *Proof of the Main Theorem:*

Let G be a group, such that nse (G) = nse(J_1) = {1,1463,5852,11704,15960,23408,25080,27720,29260, 35112}. At first we prove that $\pi(G) \subseteq \{2,3,5,7,11,19\}$. Since $1463 \in$nse(G), it follows that by $(*)$, $2 \in \pi(G)$ and $m_2 = 1463$. Let $2 \neq p \in \pi(G)$, by $(*)$, $p|(1+m_p)$ and $(p-1)|m_p$, which implies that $p \in \{3,5,7,11,13,19\}$. If $13 \in \pi_e(G)$, then by $(*)$, $m_{13} = 35112$. On the other hand, by $(*)$, we conclude that if $26 \in \pi_e(G)$, then $m_{26} \in \{15960, 25080, 27720, 35112\}$. Since $26|(1+m_2+m_{13}+m_{26})$, then $26|52536$, $26|61656$, $26|64296$ or $26|71688$, which is a contradiction. That is $26 \notin \pi_e(G)$. Thus P_{13} acts fixed point freely on the set of elements of order 2, and $|P_{13}||m_2$, which is a contradiction. Hence $13 \notin \pi_e(G)$. Therefore $\pi(G) \subseteq \{2,3,5,7,11,19\}$. If 3, 5, 7, 11 and $19 \in \pi_e(G)$, then $m_3 \in \{5852, 23408\}$, $m_5 = 11704$, $m_7 = 25080$, $m_{11} = 15960$ and $m_{19} = 27720$, by $(*)$. Also we can see easily that G does not contain any elements of order 2^6, 3^3, 5^3, 7^2, 11^3 and 19^2. Similarly, we can see that if $2^i \in \pi_e(G)$, where $2 \leq i \leq 4$, then $m_4 \in \{5852, 11704, 15960, 23408, 25080, 27720, 29260, 35112\}$, if $32 \in \pi_e(G)$, then $m_{32} = 23408$, if $9 \in \pi_e(G)$, then $m_9 \in \{25080,27720\}$, if $25 \in \pi_e(G)$, then $m_{25} = 27720$ and if $121 \in \pi_e(G)$, then $m_{121} = 27720$. In follow, we show that $\pi(G)$ could not be $\{2\}$, $3 \in \pi_e(G)$ and $\pi(G)$ must be equal to $\{2,3,5,7,11,19\}$.

Let $\pi(G) = \{2\}$. Since $2^6 \notin \pi_e(G)$, then $\pi_e(G) \subseteq \{1,2,2^2,\dots,2^5\}$. On the other hand, nse$(G)$ has ten elements, which is a contradiction.

We know that $2 \in \pi(G)$. We claim that $3 \in \pi(G)$. Suppose that $3 \notin \pi(G)$. If $5,7,11$ and $19 \notin \pi(G)$, then $\pi(G) = \{2\}$, which is a contradiction. Hence 5, 7, 11 or $19 \in \pi(G)$. If $11 \in \pi(G)$, since $11^3 \notin \pi_e(G)$, then $exp(P_{11}) = 11$ or 121. Suppose that $exp(P_{11}) = 11$. By Lemma 2.1, by considering $m = |P_{11}|$, we have $|P_{11}||(1+m_{11}) = 15961$. Hence $|P_{11}| = 11$ and so $n_{11} = m_{11}/\varphi(11) = 15960/10 = 1596$. We know that $n_{11}||G|$, since $3 \notin \pi(G)$ we get a contradiction. Now suppose that $exp(P_{11}) = 121$. We have $|P_{11}||(1+m_{11}+m_{121})$, and so $|P_{11}||(1+15960+27720)$. Hence $|P_{11}| = 121$ and $n_{11} = m_{121}/\varphi(121) = 27720/110 = 252$, since $3 \notin \pi(G)$, we get a contradiction. Therefore $11 \notin \pi(G)$. If $5 \in \pi_e(G)$, since $125 \notin \pi_e(G)$, then $exp(P_5) = 5$ or 25. If $exp(P_5) = 5$, then $|P_5||(1+m_5) = 11705$, by Lemma 2.1. Hence $|P_5| = 5$ and $n_5 = m_5/\varphi(5) = 11704/4 = 2926$. So $11 \in \pi(G)$, which is a contradiction. Similarly if $exp(P_5) = 25$, then $11 \in \pi(G)$. Thus $5 \notin \pi_e(G)$. If $7 \in \pi_e(G)$, since $49 \notin \pi_e(G)$, then $exp(P_7) = 7$. We can see easily $|P_7| = 7$ and $n_7 = 4180$. Thus $11 \in \pi(G)$, which is a contradiction. Also, if $19 \in \pi_e(G)$, then $11 \in \pi(G)$, which is a contradiction. Therefore $3 \in \pi(G)$.

Let $\pi(G) = \{2,3\}$. Since $3^3 \notin \pi_e(G)$, then $exp(P_3) = 3$ or 9. At first, suppose that $exp(P_3) = 3$. We have $|P_3||(1+m_3)$, where $m_3 \in \{5852, 23408\}$. If $m_3 = 5852$, then $|P_3| = 3$ and $n_3 = m_3/\varphi(3) = 2926$. Then $7||G|$, which is a contradiction. If $m_3 = 23408$, then $|P_3||(1+m_3) = 23409$. So $|P_3||3^4$. We consider all possibility cases.

<u>Case1</u>. If $|P_3| = 3$, then $n_3 = m_3/\varphi(3) = 11704$. Since $7 \notin \pi_e(G)$, we get a contradiction.

<u>Case</u>2. If $|P_3| = 9$, then $|G| = 2^m \times 9 = 175560 + 5852k_1 + 11704k_2 + 15960k_3 + 23408k_4 + 25080k_5 + 27720k_6 + 29260k_7 + 35112k_8$, where m, k_1, k_2, k_3, k_4, k_5, k_6, k_7, and k_8 are non-negative integers. Since $2^6 \notin \pi_e(G)$ and $exp(P_3) = 3$, then $\pi_e(G) \subseteq \{1,2,3,2^2,\dots,2^5\} \cup \{2\times 3, 2^2 \times 3,\dots,2^5 \times 3\}$. Hence $|\pi_e(G)| \leq 12$ and $0 \leq k_1+k_2+k_3+k_4+k_5+k_6+k_7+k_8 \leq 2$. By above equation we have $175560 \leq 2^m \times 9 \leq 175560 + 2 \times 35512$. So $19506 \leq 2^m \leq 27309$. Since m is a non-negative integer, then there is no possibility for m. Then we get a contradiction.

<u>Case</u>3. If $|P_3| = 27$, then $|G| = 2^m \times 27 = 175560 + 5852k_1 + 11704k_2 + 15960k_3 + 23408k_4 + 25080k_5 + 27720k_6 + 29260k_7 + 35112k_8$, where m, k_1, k_2, k_3, k_4, k_5, k_6, k_7, and k_8 are non-negative integers. As above we have $|\pi_e(G)| \leq 12$ and $0 \leq k_1+k_2+k_3+k_4+k_5+k_6+k_7+k_8 \leq 2$. Hence $175560 \leq 2^m \times 27 \leq 175560 + 2 \times 35512$. So $6502 \leq 2^m \leq 9103$. The only possibility for m is 13. If m = 13, then $|G| = 2^{13} \times 27 = 175560 + 5852k_1 + 11704k_2 + 15960k_3 + 23408k_4 + 25080k_5 + 27720k_6 + 29260k_7 + 35112k_8$, where

$0 \leq k_1+k_2+k_3+k_4+k_5+k_6+k_7+k_8 \leq 2$. Hence we have $5852k_1 + 11704k_2 + 15960k_3 + 23408k_4 + 25080k_5 + 27720k_6 + 29260k_7 + 35112k_8 = 45624$, where $0 \leq k_1+k_2+k_3+k_4+k_5+k_6+k_7+k_8 \leq 2$. It is easy to check that this equation has no solution, which is a contradiction.

Case4. If $|P_3| = 81$, then $|G| = 2^m \times 81 = 175560 + 5852k_1 + 11704k_2 + 15960k_3 + 23408k_4 + 25080k_5 + 27720k_6 + 29260k_7 + 35112k_8$, where m, k_1, k_2, k_3, k_4, k_5, k_6, k_7, and k_8 are non-negative integers. Again as above $|\pi_e(G)| \leq 12$ and $0 \leq k_1+k_2+k_3+k_4+k_5+k_6+k_7+k_8 \leq 2$. Hence $175560 \leq 2^m \times 81 \leq 175560 + 2 \times 35512$. So $2167 \leq 2^m \leq 3034$. Since m is a non-negative integer, then there is no possibility for m. Then we get a contradiction.

Now Suppose that $exp(P_3) = 9$. We have $|P_3||(1+m_3+m_9)$, where $m_3 \in \{5852, 23408\}$ and $m_9 \in \{25080, 27720\}$. It is easy to see $|P_3||9$. Since $exp(P_3)$ =9. then $|P_3| = 9$. On the other hand, we have $n_3 = m_9/\varphi(9)$. So $n_3 = 4180$ or 4620. Since $11|n_3$ and $n_3||G|$, we get a contradiction. Therefore $\pi(G)$ can not be equal to {2,3}.

Until now, we prove that $\{2,3\} \subset \pi(G) \subseteq \{2,3,5,7,11,19\}$. So at least on of the numbers 5,7,11 or 19 belong to $\pi(G)$. If $7 \in \pi(G)$, then $n_7 = 4180$. Hence $5 \in \pi(G)$. Similarly if $19 \in \pi(G)$, then $5 \in \pi(G)$. If $11 \in \pi(G)$, then $n_{11} = 252$ or 1596. Hence $7 \in \pi(G)$ and so $5 \in \pi(G)$. Therfore in any cases we can assume that $5 \in \pi(G)$. If $5 \in \pi(G)$, then $n_5 = 1386$ or 2926. Since $n_5||G|$, then $\pi(G) = \{2,3,5,7,11\}$ or $\{2,3,5,7,11,19\}$. If $\pi(G) = \{2,3,5,7,11\}$, then $n_7 = 4180$. So $19||G|$, which is a contradiction. Hence we have $\pi(G) = \{2,3,5,7,11,19\}$. Since $7^2 \notin \pi(G)$ and $19^2 \notin \pi(G)$, we have $|P_7| = 7$ and $|P_{19}| = 19$. We prove that 21, 95 and $209 \notin \pi_e(G)$. Suppose that $95 \in \pi_e(G)$, we know that if P and Q are Sylow 19 −subgroups of G, then P and Q are conjugate, which implies that $C_G(P)$ and $C_G(Q)$ are conjugate in G. Therefore $m_{95} = \varphi(95). n_{19}.k$, where k is the number of cyclic subgroups of order 5 in $C_G(P_{19})$. Since $n_{19} = m_{19}/\varphi(19) = 1540$, we have $72 \times 1540|m_{95}$, which is a contradiction. Hence $95 \notin \pi_e(G)$. Similarly we can prove that 21 and $209 \notin \pi_e(G)$. Since $95 \notin \pi_e(G)$, then the P_5 acts fixed point freely on the set of elements of order 19, and so $|P_5||m_{19} = 27720$, which implies that $|P_5| = 5$. Since $21 \notin \pi_e(G)$, the group P_3 acts fixed point freely on the set of elements of order 7 and so $|P_3||m_7 = 25080$, which implies that $|P_3| = 3$. Similarly, since $209 \notin \pi_e(G)$ we have $|P_{11}| = 11$. Suppose that $14 \in \pi_e(G)$, then $m_{14} = 25080$. Since $14 \mid (1 + m_2 + m_7 + m_{14})$, then have $14|51624$, which is a contradiction. So $14 \notin \pi_e(G)$. Thus P_2 acts fixed point freely on the set of elements of order 7. Hence $|P_2||m_7 = 25080$ and so $|P_2|\,|2^3$. On the other hand, we have $175560 \leq |G| = 2^m \times 3 \times 5 \times 7 \times 11 \times 19$, where $m \leq 3$. Therefore $|G| = |J_1| = 175560$.

We know that $m_{19}(G) = 18k$, where k is the number of cyclic subgroups of order 19. Since $nse(G) = nse(J_1)$, we can see easily $m_{19}(G) = m_{19}(J_1)$. On the other hand, since $19^2 \nmid |G|$, then P_{19} is cyclic and k, is the number of Sylow 19 −subgroups of G. Hence $n_{19}(G) = n_{19}(J_1)$. Now by Lemma 2.2, since $|G| = |J_1|$ and $n_{19}(G) = n_{19}(J_1)$, we can conclude $G \cong J_1$ and the proof is completed.

Problem:

Is the sporadic simple group *S*, characterizable by nse(S)?

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