Australian Journal of Basic and Applied Sciences, 6(1): 198-206, 2012
ISSN 1991-8178

Estimation of Parameters of the Exponential Geometric Distribution with Presence of Outliers Generated from Uniform Distribution

Parviz Nasiri, Ali Shadrokh and Hassan Pazira

Department of Statistics, University of Payame Noor, Tehran Branch, Tehran, Iran

Abstract: A two-parameter distribution with decreasing failure is introduced with presence of outliers generated from uniform distribution. Various properties are discussed and the estimation of parameters is studied by the methods of moment and maximum likelihood. These estimators are compared empirically when all of the parameters are unknown. Their bias and mean square error (MSE) are investigated with help of numerical techniques.

Key words: Compounding, Exponential Geometric distribution, Failure rate, Uniform distribution, Moment estimation, Maximum likelihood estimation, Newton-Raphson Method, Monte-Carlo Simulation.

INTRODUCTION

The study of length life of organisms, devices, structures, materials, etc., is of major importance in the biological and engineering sciences. A substantial part of such study is devoted to the mathematical description of the length of life by a failure distribution. Sometimes physical considerations of the failure mechanism may lead to a specific distribution but more often, the choice is made on the basis of how well the actual observations of times to failure appear to be fitted by distribution.

Situations where the failure rate function decreases with time have been reported by several authors. Indicative examples are the business mortality Lomax (1954), failures in the air-conditioning equipment of a fleet of Boeing 720 aircrafts of in semiconductors from various combined Proschan (1963) and the life of integrated circuit modules Saunders and Myher (1983). In general, a population is expected to exhibit decreasing failure rate (DFR) when its behavior over time is characterized by work hardening (in engineering terms) or immunity (in biological terms); sometimes the broader term 'infant mortality' is used to denote the DFR phenomenon. Proschan (1963) proved that the DFR property is inherent to mixtures of distributions with constant failure rate and Gleser (1989) demonstrated the converse for any gamma distribution with shape parameter less than one.

According to Dixit, Moore and Barnett (1996), we assume that a set of random variables ($X_1, X_2,..., X_n$) represent the distance of an infected sample plant from a plot of plants inoculated with a virus. Some of the observations are derived from the airborne dispersal of the spores and are distributed according to the exponential distribution. The other observations out of n random variables (say k) are present because aphids which are know to be carries of barley yellow mosaic dwarf virus (BYMDV) have passed the virus into the plants when the aphids feed on the sap. These k (know) aphids are considered to be exponential distributed.

Thus, we assume that the random variables ($X_1, X_2,..., X_n$) are such that k of them are distributed with p.d.f $g(x,\theta)$,

$$g(x,\theta)=\frac{1}{\theta} \ , \ \ 0<x<\theta \tag{1.1}$$

and the remaining (n–k) random variables are distributed with p.d.f $f(x, \theta, p)$,

$$f(x,\theta,p)=\theta(1-p)\,e^{-\theta x}(1-p\,e^{-\theta x})^{-2} \ , \qquad x>0 \tag{1.2}$$

In section 2, we introduce the exponential geometric distribution, and in section 3, we have obtained the joint Distribution of ($X_1, X_2,..., X_n$) in the presence of k outliers generated from uniform distribution, also in this section we present the survival failure rate function. In sections 4 and 5, we deal with estimation

Corresponding Author: Parviz Nasiri, Department of Statistics, University of Payame Noor, Tehran Branch, Tehran, Iran.

parameters by using the methods of the moment and maximum likelihood, respectively. In section 6, we have given a numerical study and conclusion.

Exponential Geometric Distribution:

Suppose that $Y_1, Y_2, \ldots, Y_z$ are iid with density $f(x, \theta)$.

$$f(y,\theta) = \theta e^{-\theta y} \quad , \quad y > 0 \ , \quad \theta > 0 \tag{2.1}$$

and Z is a geometric variable with probability function $P[Z = z]$;

$$P(Z = z) = (1-p)p^{z-1} \quad , \quad z = 1,2,\ldots \quad , \quad 0 < p < 1 \tag{2.2}$$

If we consider the random variable

$$X = min(X_1, X_2, \ldots, X_z)$$

then

$$f(x,\theta,p) = \theta(1-p)e^{-\theta x}(1-pe^{-\theta x})^{-2} \quad , \quad x > 0 \tag{2.3}$$

Proof: We know that

$$f(x\,|\,z,\theta) = \frac{n!}{k!(n-k)!}F^{k-1}(x\,|\,z)\left(1-F(x\,|\,z)\right)^{n-k} f(x\,|\,z)$$

$$= z\theta e^{-(z-1)\theta x}e^{-\theta x} = z\theta e^{-z\theta x}$$

then

$$\begin{aligned} f(x\,|\,\theta) &= \sum_{z=1}^{\infty} f(x,z\,|\,\theta) = \sum_{z=1}^{\infty} f(x\,|\,z,\theta)P(Z=z) \\ &= \theta(1-p)\sum_{z=1}^{\infty} z\,p^{z-1}e^{-z\theta x} \\ &= \theta(1-p)e^{-\theta x}\sum_{z=1}^{\infty} z\left(pe^{-\theta x}\right)^{z-1} = \theta(1-p)\sum_{z=1}^{\infty}\frac{d}{dp}\left(pe^{-\theta x}\right)^{z} \\ &= \theta(1-p)e^{-\theta x}\left(1-pe^{-\theta x}\right)^{-2} \quad , \quad x > 0 \ , \ 0 < p < 1 \end{aligned}$$

The latter defines the distribution that we shall be referring to in the sequel as the exponential geometric distribution (EG for brevity).

It can be verified by standard techniques that if X is an EG variable, with density given by (2.3) then, the random variable $y = p^{-1}(e^{\theta x} - 1)$ follows the Pareto distribution, with shape and scale parameters one and $P(1-P)^{-1}$, respectively.

Joint Distribution of ($X_1, X_2, \ldots, X_n$) with κ Outliers:

According to Dixit (1989) model, we assume that the random variables ($X_1, X_2, \ldots, X_n$) are such that k of them are (1.1) and remaining (n-k) random variables are distribution as (1.2). The joint distribution of ($X_1, X_2, \ldots, X_n$) is given as

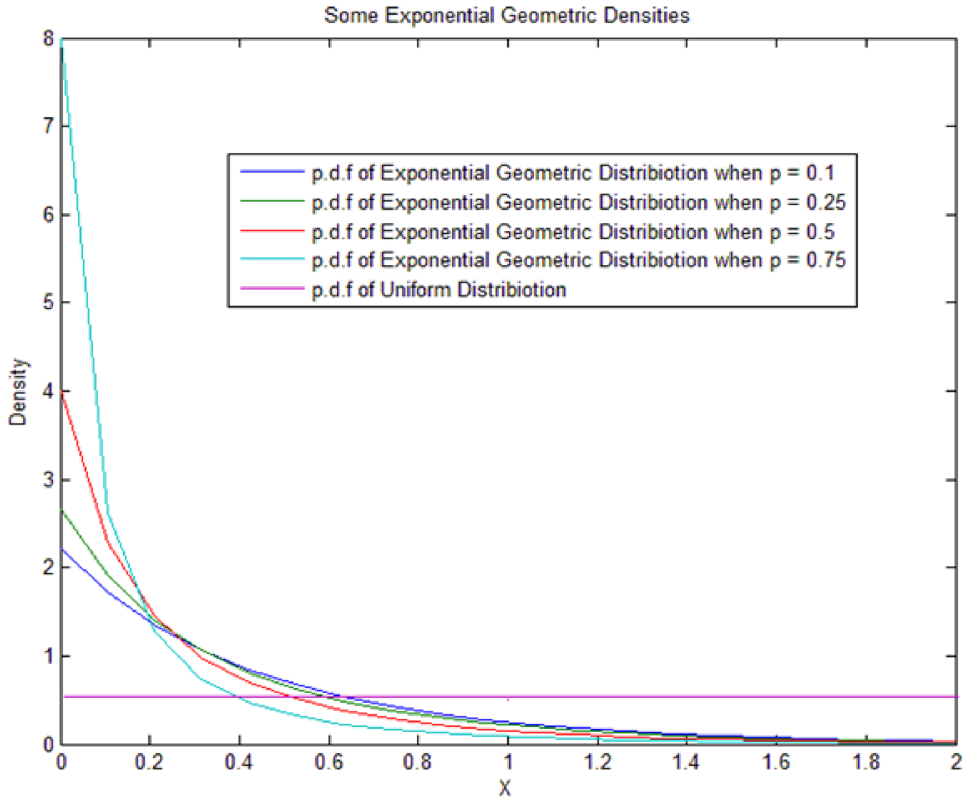

Fig. 1: p.d.f. of the Exponential Geometric and Uniform Distribution for different values of p and θ = 2.

$$f(x_1,x_2,...,x_n|\theta,p)=\frac{k!(n-k)!}{n!}\prod_{i=1}^{n} f(x_i|\theta,p)\sum_{\underline{A}}\prod_{r=1}^{k}\frac{g(x_{A_r}|\theta)}{f(x_{A_r}|\theta,p)}$$

$$=\frac{k!(n-k)!}{n!}\theta^n(1-p)^n e^{-\theta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}(1-p\,e^{-\theta x_i})^{-2}$$

$$\times\sum_{\underline{A}}\frac{\prod_{r=1}^{k} I_{(0,\theta)}(x_{A_r})}{\theta^{2k}(1-p)^k\prod_{r=1}^{k}e^{-\theta x_{A_r}}(1-p\,e^{-\theta x_{A_r}})^{-2}}$$

$$=\frac{k!(n-k)!}{n!}\theta^{n-2k}(1-p)^{n-k}e^{-n\theta\bar{x}}\prod_{i=1}^{n}(1-p\,e^{-\theta x_i})^{-2}$$

$$\times\sum_{\underline{A}}\frac{\prod_{r=1}^{k} I_{(0,\theta)}(x_{A_r})}{\prod_{r=1}^{k}e^{-\theta x_{A_r}}(1-p\,e^{-\theta x_{A_r}})^{-2}} \qquad (3.1)$$

where $\sum_{\underline{A}}=\sum_{A_1=1}^{n-k+1}\sum_{A_2=A_1+1}^{n-k+2}\cdots\sum_{A_k=A_{k-1}+1}^{n}$ and $k=1,2,3,....,n$. For the more details see Dixit *et al*. (1996), Dixit and Nasiri (2001) and Nasiri and Pazira (2010). For *k=0,* it is given by Adamidis and Loukas (1998). From (3.1) the marginal distribution of X is

$$f(x,\theta,p)=\frac{k}{n}\theta e^{-\theta x}+\frac{n-k}{n}\theta(1-p)e^{-\theta x}(1-pe^{-\theta x})^{-2} \quad , \qquad x>0 \tag{3.2}$$

From (3.2) the survival probability is given by

$$\begin{aligned} s(x,\theta,p) &= 1-F(x,\theta,p)=1-\int_0^x f(t,\theta,p)\,dt \\ &= 1-\int_0^x \frac{k}{n}\theta e^{-\theta t}\,dt-\int_0^x \frac{n-k}{n}\theta(1-p)e^{-\theta t}(1-pe^{-\theta t})^{-2}\,dt \end{aligned} \tag{3.3}$$

The failure rate (also known as hazard rate) is

$$h(x)=\frac{f(x,\theta,p)}{s(x,\theta,p)}=\frac{\frac{k}{n}\theta e^{-\theta x}+\frac{n-k}{n}\theta(1-p)e^{-\theta x}(1-pe^{-\theta x})^{-2}}{1-\frac{k}{n}(1-e^{-\theta x})-\frac{n-k}{n}\left[p^{-1}-(1-p)p^{-1}(1-pe^{-\theta x})^{-1}\right]} \tag{3.4}$$

Method of Moment:

The raw moments of X may be determined from (3.2) be direct integration. For $r\in \mathrm{N}$, we fine that

$$\begin{aligned} E(X^r) &= \int_0^\infty x^r\left[\frac{k}{n}\theta e^{-\theta x}+\frac{n-k}{n}\theta(1-p)e^{-\theta x}(1-pe^{-\theta x})^{-2}\right]dx \\ &= \int_0^\infty \frac{k}{n}x^r\,\theta e^{-\theta x}\,dx+\int_0^\infty \frac{n-k}{n}x^r\,\theta(1-p)e^{-\theta x}(1-pe^{-\theta x})^{-2}\,dx \\ &= \frac{k}{n\theta^r}\Gamma(r+1)+\frac{n-k}{n}\int_0^\infty x^r\,\theta(1-p)e^{-\theta x}(1-pe^{-\theta x})^{-2}\,dx \end{aligned}$$

Let $u=x^r$ and $\theta pe^{-\theta x}(1-pe^{-\theta x})^{-2}\,dx=dv$ then

$$\begin{aligned} E(X^r) &= \frac{k}{n\theta^r}\Gamma(r+1)+\frac{n-k}{n}\int_0^\infty \frac{1-p}{p}u\,dv \\ &= \frac{k}{n\theta^r}\Gamma(r+1)+\frac{n-k}{n}\frac{1-p}{p}\left[uv|_0^\infty-\int_0^\infty v\,du\right] \\ &= \frac{k}{n\theta^r}\Gamma(r+1)+\frac{n-k}{n}\frac{1-p}{p}\left[\frac{-x^r}{1-pe^{-\theta x}}|_0^\infty-\int_0^\infty \frac{r\,x^{r-1}}{1-pe^{-\theta x}}dx\right] \\ &= \frac{k}{n\theta^r}\Gamma(r+1)+\frac{n-k}{n}\frac{1-p}{p}\int_0^\infty \frac{r\,x^{r-1}}{1-pe^{-\theta x}}dx \end{aligned}$$

Since $\left|pe^{-\theta x}\right|<1$

$$\frac{1}{1-p\,e^{-\theta x}}=\sum_{i=0}^{\infty}(p\,e^{-\theta x})^{i} \tag{4.1}$$

Hence

$$E(X^{r})=\frac{k}{n\theta^{r}}\Gamma(r+1)+\frac{n-k}{n}\frac{1-p}{p}\int_{0}^{\infty}r\,x^{r-1}\sum_{i=0}^{\infty}(p\,e^{-\theta x})^{i}\,dx$$

$$=\frac{k}{n\theta^{r}}\Gamma(r+1)+\frac{n-k}{n}\frac{1-p}{p}\sum_{i=0}^{\infty}\int_{0}^{\infty}r\,x^{r-1}(p\,e^{-\theta x})^{i}\,dx$$

$$=\frac{k}{n\theta^{r}}\Gamma(r+1)+\frac{n-k}{n}\frac{1-p}{p}\sum_{i=1}^{\infty}p^{i}(\theta i)^{-r}r!$$

$$=\frac{\Gamma(r+1)}{n\theta^{r}}\left[k+(n-k)(1-p)\sum_{i=1}^{\infty}\frac{p^{i-1}}{i^{r}}\right] \tag{4.2}$$

For $r=1$ and 2, $E(X^{r})$ is given by

$$E(X)=\frac{1}{n\theta}\left[k+(n-k)(1-p)\sum_{i=1}^{\infty}\frac{p^{i-1}}{i}\right] \tag{4.3}$$

$$E(X^{2})=\frac{2}{n\theta^{2}}\left[k+(n-k)(1-p)\sum_{i=1}^{\infty}\frac{p^{i-1}}{i^{2}}\right] \tag{4.4}$$

Let $D=\frac{m_1'^{2}}{m_2'}$, where $m_i'=\frac{1}{n}\sum_{j=1}^{n}X_j^{i}$. Then from (4.3) and (4.4) we have

$$D=\frac{\left(k+(n-k)(1-\hat{p})\sum_{i=1}^{\infty}\frac{\hat{p}^{i-1}}{i}\right)^{2}}{2n\left(k+(n-k)(1-\hat{p})\sum_{i=1}^{\infty}\frac{\hat{p}^{i-1}}{i^{2}}\right)} \tag{4.5}$$

where $\hat{p}$ is estimate of p. Next, from (4.3) we can obtain estimate of θ as following

$$\hat{\theta}=\frac{1}{n\,m_1'}\left(k+(n-k)(1-\hat{p})\sum_{i=1}^{\infty}\hat{p}^{i-1}/i\right) \tag{4.6}$$

where $\hat{p}$ is given by (4.5).

Method of Maximum Likelihood:

One sees from (4.2) that moment estimates for the parameters of the EG distribution with density function cannot be obtained in closed forms and therefore that is little point in considering the method any further. Proceeding with the method of maximum likelihood, the likelihood function from a sample of *n* observations, (X_1, X_2,..., X_n) is given by

$$L(\theta,p)=\frac{k!(n-k)!}{n!}\theta^{n-2k}(1-p)^{n-k}e^{-n\theta\bar{x}}\prod_{i=1}^{n}(1-p\,e^{-\theta x_i})^{-2}$$
$$\times\sum_{\underline{A}}\frac{\prod_{r=1}^{k}I_{(0,\theta)}(x_{A_r})}{\prod_{r=1}^{k}e^{-\theta x_{A_r}}(1-p\,e^{-\theta x_{A_r}})^{-2}} \tag{5.1}$$

then

$$L(\theta,p)\approx(x_{(n)})^{n-2k}(1-p)^{n-k}e^{-n\hat{\theta}\bar{x}}\prod_{i=1}^{n}(1-p\,e^{-\hat{\theta}x_i})^{-2}$$
$$\times\sum_{\underline{A}}\prod_{r=1}^{k}e^{\hat{\theta}x_{A_r}}(1-p\,e^{-\hat{\theta}x_{A_r}})^{2} \tag{5.2}$$

where

$$\hat{\theta}=X_{(n)}=\max(X_1,X_2,...,X_n) \tag{5.3}$$

To estimate *p*, we consider $\ln\left(L(\hat{\theta},p)\right)$ as

$$\ln\left(L(\hat{\theta},p)\right)\approx(n-2k)\ln(x_{(n)})+(n-k)\ln(1-p)-\hat{\theta}\sum_{i=1}^{n}x_i-2\sum_{i=1}^{n}\ln(1-p\,e^{-\hat{\theta}x_i})$$
$$+\ln\left(\sum_{\underline{A}}\prod_{r=1}^{k}e^{\hat{\theta}x_{A_r}}(1-p\,e^{-\hat{\theta}x_{A_r}})^{2}\right) \tag{5.4}$$

Taking the derivative with respect to *p* and equating to 0, we obtain the normal equation as

$$\frac{dL(p)}{dp}\approx-\frac{(n-k)}{(1-p)}+2\sum_{i=1}^{n}\frac{e^{-\theta x_i}}{(1-p\,e^{-\theta x_i})}-\frac{2\sum_{\underline{A}}\prod_{r=1}^{k}(1-p\,e^{-\hat{\theta}x_{A_r}})}{\sum_{\underline{A}}\prod_{r=1}^{k}e^{\theta x_{A_r}}(1-p\,e^{-\theta x_{A_r}})^{2}} \tag{5.5}$$

Since $\frac{dL(p)}{dp}=0$, hence

$$\sum_{i=1}^{n}\frac{e^{-\hat{\theta}x_i}}{(1-p\,e^{-\hat{\theta}x_i})}=\frac{(n-k)}{2(1-p)}+\frac{\sum_{\underline{A}}\prod_{r=1}^{k}(1-p\,e^{-\hat{\theta}x_{A_r}})}{\sum_{\underline{A}}\prod_{r=1}^{k}e^{\hat{\theta}x_{A_r}}(1-p\,e^{-\hat{\theta}x_{A_r}})^2} \tag{5.6}$$

Here, we need to use either the scoring algorithm or the Newton-Raphson method to solve the non-linear equation. Here, we solve (5.6) by Newton-Raphson method. Hence solution of the equation is

$$p_{i+1}=p_i-\frac{g(p_i)}{g'(p_i)},\qquad i=0,\,1,\,2,\,... \tag{5.7}$$

where

$$g(p_i)=\sum_{i=1}^{n}\frac{e^{-\hat{\theta}x_i}}{(1-p\,e^{-\hat{\theta}x_i})}-\frac{(n-k)}{2(1-p)}-\frac{\sum_{\underline{A}}\prod_{r=1}^{k}(1-p\,e^{-\hat{\theta}x_{A_r}})}{\sum_{\underline{A}}\prod_{r=1}^{k}e^{\hat{\theta}x_{A_r}}(1-p\,e^{-\hat{\theta}x_{A_r}})^2} \tag{5.8}$$

and

$$g'(p_i)=\sum_{i=1}^{n}\frac{e^{-2\hat{\theta}x_i}}{(1-p\,e^{-\hat{\theta}x_i})^2}-\frac{(n-k)}{2(1-p)^2}+\frac{\sum_{\underline{A}}\prod_{r=1}^{k}e^{-\hat{\theta}x_{A_r}}}{\sum_{\underline{A}}\prod_{r=1}^{k}e^{\hat{\theta}x_{A_r}}(1-p\,e^{-\hat{\theta}x_{A_r}})^2}-2\left(\frac{\sum_{\underline{A}}\prod_{r=1}^{k}(1-p\,e^{-\hat{\theta}x_{A_r}})}{\sum_{\underline{A}}\prod_{r=1}^{k}e^{\hat{\theta}x_{A_r}}(1-p\,e^{-\hat{\theta}x_{A_r}})^2}\right)^2 \tag{5.9}$$

Here, the initial solution p_0 should be selected from (4.5).

Numerical Study:

In this paper, we have addressed the problem of estimating parameters of Exponential Geometric distribution in presence of *k* outliers. In order to have some idea about Bias and Mean Square Error (MSE) of methods of moment and MLE, we perform sampling experiments using a MATLAB. The results are given in Tables 1 to 3, for n = 10(5) 40 and 50, p = 0.5, θ = 3, and =1, 2 and 3. We report the average estimates and the MSEs based on *1000* replications.

From Tables 1 to 3, we conjecture that the moment (MOM) and maximum likelihood (MLE) estimators of *p* are underestimation, but the moment and maximum likelihood estimators of θ are overestimation, this is true for *k* = 1,2 and 3. According to Table 1 to 3, when *n* increase then the MSEs decrease, this is true for *k* =1, 2 and 3, but when *k* increase then MSEs increase. Tables are shown that the MSE of MLE estimators of θ are less than the MSE of the MOM estimators for all values of *n* and *k*. Also, the MSE of MLE estimators of *p* are less than the MSE of the MOM estimators for all values of *n*, but only for *k* > 1. Indeed, for *k* = 1 the MSE of MOM estimators of *p* are less than the MSE of the MLE estimators for all values of *n.* So the MLE estimators of *p* and θ are more efficient than the MOM estimators. We strongly feel MLE estimators are better than the MOM estimations. From the previous observations, we suggest to use MLE method for estimating *p* and θ in Exponential Geometric distribution with presence of *k* outliers.

Table 1: p = 0.5, θ = 3 and k =1.

n	Method	Bias $\hat{p}$	MSE $\hat{p}$	Bias $\hat{\theta}$	MSE $\hat{\theta}$
10	MOM	0.1597	0.0954	-1.3711	2.9051
	MLE	0.2226	0.1002	-1.3749	1.1187
15	MOM	0.1087	0.0823	-0.9750	2.3970
	MLE	0.2177	0.0925	-1.3192	1.0056
20	MOM	0.1760	0.0764	-1.0142	2.2895
	MLE	0.2145	0.0823	-1.3358	0.9932
25	MOM	0.1865	0.0738	-0.7255	1.9888
	MLE	0.2249	0.0755	-1.2991	0.8401
30	MOM	0.1417	0.0707	-0.8936	1.7755
	MLE	0.2492	0.0710	-1.3119	0.8249
35	MOM	0.0651	0.0664	-0.5871	1.3725
	MLE	0.2201	0.0683	-1.1338	0.8237
40	MOM	0.1092	0.0524	-0.7284	1.3555
	MLE	0.2285	0.0588	-1.2023	0.7899
50	MOM	0.0819	0.0421	-0.5438	1.3397
	MLE	0.2020	0.0441	-1.1790	0.6604

Table 2: p = 0.5, θ = 3 and k =1.

n	Method	Bias $\hat{p}$	MSE $\hat{p}$	Bias $\hat{\theta}$	MSE $\hat{\theta}$
10	MOM	0.1557	0.0957	-1.6609	3.2675
	MLE	0.2024	0.0790	-0.9430	1.5572
15	MOM	0.1515	0.0906	-1.3743	2.6384
	MLE	0.1982	0.0788	-0.9968	1.4985
20	MOM	0.1892	0.0852	-1.3481	2.5466
	MLE	0.1958	0.0785	-1.0307	1.4010
25	MOM	0.1835	0.0833	-1.2813	2.4723
	MLE	0.1921	0.0730	-0.9318	1.2399
30	MOM	0.2046	0.0823	-1.3128	2.2906
	MLE	0.1815	0.0620	-0.8976	1.1389
35	MOM	0.2087	0.0789	-1.2886	2.2109
	MLE	0.1684	0.0598	-0.8553	1.0664
40	MOM	0.1601	0.0778	-1.1431	1.9441
	MLE	0.1660	0.0505	-0.8739	1.0402
50	MOM	0.1783	0.0682	-1.1391	1.8377
	MLE	0.1559	0.0500	-0.8357	1.0182

Table 3: p = 0.5, θ = 3 and k =1.

n	Method	Bias $\hat{p}$	MSE $\hat{p}$	Bias $\hat{\theta}$	MSE $\hat{\theta}$
10	MOM	0.1189	0. 0969	-1.7582	3. 4270
	MLE	0. 1831	0. 0913	-0.8739	2. 6194
15	MOM	0. 1842	0. 0950	-1. 7638	3. 4377
	MLE	0. 1771	0. 0907	-0.7723	2.4575
20	MOM	0. 1914	0. 0935	-1. 5517	2. 8678
	MLE	0. 1429	0. 0892	-0.7052	2.3268
25	MOM	0. 2311	0. 0927	-1. 5455	2. 7503
	MLE	0. 1397	0. 0875	-0.7321	2.2334
30	MOM	0. 1924	0. 0889	-1. 3534	2. 5552
	MLE	0. 1396	0. 0860	-0.8187	2.1696
35	MOM	0. 2313	0. 0865	-1. 4513	2. 4950
	MLE	0. 1001	0. 0562	-0.7514	2.0038
40	MOM	0. 2423	0. 0815	-1. 4504	2. 3638
	MLE	0. 1148	0. 0559	-0.6628	1.8106
50	MOM	0. 2113	0. 0776	-1. 3028	2. 0112
	MLE	0. 1001	0. 0519	-0.7186	1.8603

REFERENCES

Adamidis, K. and S. Loukas, 1998. A lifetime distribution with deceasing failure rate. Statistics and probability letter 39, 35-41.

Dixit, U.J., 1989. Estimation of parameters of the gamma distribution in the presence of outliers. Communications in statistics – theory and methods, 18: 3071-3085.

Dixit, U.J., Moor, K.L. and V. Barnett, 1996. On the estimation of the power of the scale parameter of the exponential distribution in the presence of outliers generated from uniform distribution, Metron, 54: 201-211.

Dixit, U.J. and P.F. Nasiri, 2001. Estimation of parameters of the exponential distribution in the presence of outliers generated from uniform distribution, Metron, 49(3-4): 187-198.

Glesser, L.J., 1989. The gamma distribution as a mixture of exponential distributions. Amer. Statist., 43: 115-117.

Lomax, K.S., 1954. Business failures: another example of the analysis of failure data. J. Amer. Statist. Assoc., 49: 847-852.

Nasiri, P.F. and H. Pazira, 2010. Bayesian and non-Bayesian estimations on the Generalized Exponential distribution in the presence outliers, Journal of Statistical Theory and Practice, 4(3): 453-475.

Proschan, F., 1963. Theoretical explanation of observed decreasing failure rate. Technometrics, 5: 375-363.

Saunders, S.C. and J.M. Myher, 1983. Maximum likelihood estimation for two-parameter decreasing hazard rate distribution using censored data. J. Amer. Statist. Assoc., 78: 664-673.