Australian Journal of Basic and Applied Sciences, 6(6): 331-336, 2012
ISSN 1991-8178

A Simple Method to Solve Quartic Equations

[1]Amir Fathi, [2]Pooya Mobadersany, [3]Rahim Fathi

[1]Department of Electrical Engineering, Urmia branch, Islamic Azad University, Urmia, Iran
[2]Department of Electrical and Computer Engineering, University of Tabriz, Tabriz, Iran
[3]Department of Electrical Engineering, Urmia branch, Islamic Azad University, Urmia, Iran

Abstract: Polynomials of high degrees often appear in many problems such as optimization problems. Equations of the fourth degree or so called quartics are one type of these polynomials. Since now there is not any simple method to solve the general forms of quartic equations. In this paper we propose a novel, simple and precise analytical method to solve quartic equations without any constraints.

Key words: Polynomials, Quartics, Equations of the Fourth Degree

INTRODUCTION

A quartic function is a polynomial of degree four. The general form of a quartic function is as follows:

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0 \tag{1}$$

Setting $f(x) = 0$ results in a quartic equation of the form written below:

$$ax^4 + bx^3 + cx^2 + dx + e = 0, \quad a \neq 0 \tag{2}$$

The quartic is the highest order polynomial equation that can be solved by radicals in the general case (I. Stewart, 2004) (i.e., one where the coefficients can take any value). In 1540 Lodovico Ferrari discovered the solution to the quartic equation (J.J. O'Connor and E. F. Robertson). Like all algebric solutions of the quartic equations, this method required the solution of a cubic equation to be found. The general form of a cubic equation is as follows:

$$ax^3 + bx^2 + cx + d = 0, \quad a \neq 0 \tag{3}$$

Gerolamo Cardano proved that the solution of the general cubic function in equation (3) is (N. Jacobson, 2009):

$$x_1 = -\frac{b}{3a} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \tag{4}$$

Where:

$$\begin{cases} p = -\dfrac{b^2}{3a^2} + \dfrac{c}{a} \\ q = \dfrac{2b^3}{27a^3} - \dfrac{bc}{3a^2} + \dfrac{d}{a} \end{cases} \tag{5}$$

By having x_1, two other solutions for equation (3), x_2 and x_3, can easily be found.

The most important point in all of the methods for solving a quartic equation is the complexity of these solutions. Some methods assume several constraints in order to simply lead to the response. In this paper we have proposed a novel, simple and precise solution for solving all general forms of quartic equations without any constraints. In section 2 we are going to describe our simple general solution and in section 3 we will provide the solution with two examples. Finally conclusions are summarized at section 4.

Corresponding Author: Amir Fathi, Department of Electrical Engineering, Urmia branch, Islamic Azad University, Urmia, Iran
E-mail: fathi.amir@hotmail.com

2. The Proposed Simple Method to Solve The Quartics:

Assume the general form of the quartic shown in equation (6):

$$ax^4+bx^3+cx^2+dx+e=0\,,\quad a\neq 0 \tag{6}$$

Let $x=t-\dfrac{b}{4a}$

The equation (6) leads to:

$$t^4+pt^2+qt+s=0 \tag{7}$$

In which p , q and s are defined as below:

$$\begin{cases} p=-\dfrac{3b^2}{8a^2}+\dfrac{c}{a} \\ q=\dfrac{b^3}{8a^3}-\dfrac{bc}{2a^2}+\dfrac{d}{a} \\ s=-\dfrac{3b^4}{256a^4}+\dfrac{b^2c}{16a^3}-\dfrac{bd}{4a^2}+\dfrac{e}{a} \end{cases} \tag{8}$$

In order to solve the equation (7) assume equation (9) with two variables (B and D) with a relationship shown in equation (9):

$$(t^2+B)^2=(t+D)^2 \tag{9}$$

Since equation (9) can be solved easily, we can try to set the equation (7) equal to the equation (9).

Then from equation (9) we have:

$$(t^4+2Bt^2+B^2)=(t^2+2Dt+D^2)$$

Sorting the equation above leads to:

$$t^4+(2B-1)t^2-2Dt+B^2-D^2=0 \tag{10}$$

In order to set the equations (10) and (7) equal, the expressions written below should be true:

$$\begin{cases} p=2B-1 \\ q=-2D \end{cases} \Rightarrow \begin{cases} B=(p+1)/2 \\ D=-q/2 \end{cases} \tag{11}$$

And to set:

$$s=B^2-D^2 \tag{12}$$

Variable changes in equation (13) should be applied:

$$t=zy \tag{13}$$

Replacing equation (13) in equation (7) leads to:

$$z^4y^4 + pz^2y^2 + qzy + s = 0$$

$$y^4 + (\frac{p}{z^2})y^2 + (\frac{q}{z^3})y + \frac{s}{z^4} = 0$$

$$c' = \frac{p}{z^2}$$

$$d' = \frac{q}{z^3}$$

$$e' = \frac{s}{z^4}$$

$$\Rightarrow e' = B'^2 - D'^2$$

So:

$$B' = \frac{(c'+1)}{2}, D' = -\frac{d'}{2}$$

$$e' = B'^2 - D'^2 \Rightarrow \frac{s}{z^4} = [(\frac{p}{z^2}+1)^2 - (\frac{q}{z^3})^2]/4$$

$$\Rightarrow 4s = p^2 + z^4 + 2z^2p - \frac{q^2}{z^2}$$

$$\Rightarrow z^6 + 2z^4p + (p^2 - 4s)a^2 - q^2 = 0$$

$$(z^2)^3 + 2p(z^2)^2 + (p^2 - 4s)z^2 - q^2 = 0 \quad (14)$$

Replacing arbitrary one of the roots of equation (14) in equation (13) the equation (12) will be set, then the equation (10) will be equal to the equation (7). Therefore finding the roots of equation (10) from equation (9), the roots of equation (7) will be found easily.

3. Examples:

Example 1: Find the roots of $t^4 - t^2 + 2t + 2 = 0$.

Solution:

By constructing equation (14) for this example, z will be found:

$$z^6 - 2z^4 - 7z^2 - 4 = 0 \quad (15)$$

$$\Rightarrow z^2 = 4 \Rightarrow z = 2$$

$$t = zy \Rightarrow t = 2y$$

Replacing $t = 2y$ in the given equation in the example leads to:

$$\Rightarrow y^4 - 1/4y^2 + 2/8y + 2/16 = 0$$

$$\Rightarrow y^4 - 1/4y^2 + 1/4y + 1/8 = 0$$

$$B = (-1/4 + 1)/2$$

$$D = -q/2 = -1/8$$

$$(y^2 + 3/8)^2 = (y - 1/8)^2$$

Taking square root from the equation above:

$$y^2 + 3/8 = \pm(y - 1/8)$$
$$y^2 + 3/8 = +(y - 1/8)$$
$$\Rightarrow y^2 - y + 1/2 = 0$$
$$\Rightarrow y_{1,2} = (1 \pm \sqrt{(1-2)})/2 = (1 \pm i)/2$$

So two roots of the given equation in this example are:

$$t_{1,2} = 2y_{1,2} = 1 \pm i$$

In the other hand we have:

$$y^2 + 3/8 = -(y - 1/8)$$

So:

$$y^2 + y + 1/4 = 0$$
$$\Rightarrow y_{3,4} = (-1 \pm \sqrt{(1-1)})/2 = -1/2$$

So other two roots of the given equation in the example are:

$$t_{3,4} = 2y_{3,4} \Rightarrow t_{3,4} = -1$$

Note that $t_{3,4} = -1$ is a double root.

Therefore the roots of given equation is found simply as follows:

$$t_1 = 1 + i$$
$$t_2 = 1 - i$$
$$t_{3,4} = -1$$

Note that for the other five roots of equation (15) same results will be obtained.

Example 2:

Find the roots of

$$x^4 + 4x^3 + 33x^2 + (58 - 14i)x + 148 - 14i = 0.$$

Solution:

First the coefficient of y^3 is set to zero by the variable change of: $x = t - 1$

With this variable changing the given equation changes to:

$$t^4 + 27t^3 - 14it + 120 = 0 \qquad (16)$$

In this equation we have:

$$s = 120, \; q = -14i \;, \; p = 27$$

In order to find z the equation (14) should be constructed. After constructing the equation (14) and finding z then the given equation will change to the desired equation (9). The form equation (14) for this example is:

$$z^6 - 54z^4 + 249z^2 + 196 = 0 \tag{17}$$
$$\Rightarrow (z^2)^3 - 54(z^2)^2 + 249z^2 + 196 = 0$$

After solving this equation of degree 6 by simplifying its form into an equation of degree 3 we have:

$z = i$ (This is one of the solutions for equation (17))

$\Rightarrow y = it$

By making this variable change into the equation (16) we have:

$$y^4 - 27y^2 - 14y + 120 = 0$$

In this equation:

$$B = \frac{-27+1}{2} = -13$$
$$D = -\frac{(-14)}{2} = 7$$
$$\Rightarrow (y^2 - 13)^2 = (y+7)^2$$
$$\Rightarrow \begin{cases} y^2 - 13 = y + 7 \\ y^2 - 13 = -y - 7 \end{cases} \Rightarrow \begin{cases} y^2 - y - 20 = 0 \Rightarrow \begin{cases} y_1 = -4 \\ y_2 = 5 \end{cases} \\ y^2 + y - 6 = 0 \Rightarrow \begin{cases} y_3 = 2 \\ y_4 = -3 \end{cases} \end{cases}$$

So:

$$t_1 = 4i$$
$$t_2 = -5i$$
$$t_3 = -2i$$
$$t_4 = 3i$$

And the solutions for the given equation will be found as following:

$$x_1 = -1 + 4i$$
$$x_2 = -1 - 5i$$
$$x_3 = -1 - 2i$$
$$x_4 = -1 + 3i$$

Conclusions:

The most important point in all of the methods for solving a quartic equation is the complexity of these solutions. Some methods assume several constraints in order to simply lead to the response. In this paper we have proposed a novel, simple and precise solution for solving all general forms of quartic equations without any constraints. With the proposed initiated method all forms of the quartics can be solved easily. Note that there is not any approximation in the resulted answers of this method, and all of the answers are precise. In this method a quartic is transformed to an equation of degree two that can easily be solved. In the proposed method the coefficients of the resulted equation of degree two are easily found from solving an equation of degree three. To prove the efficiency and simplicity of the proposed method an example quartic is given in the third section of the paper and it is solved with the proposed initiated method.

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