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Uniqueness and Existence of Solutions of the Periodic First-Order Fuzzy Differential Equation with Boundary Value Using Fixed Point Theorems

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Abstract: In this paper, the existence and uniqueness of solution of the periodic first-order fuzzy differential equation with boundary value using fixed point theorem are discussed. To do these, Minimal and maximal solutions are defined and some theorems are proved in detail.

Key words: Fuzzy differential equation; Fixed point; Minimal and maximal solutions; Existence and uniqueness of solution.

INTRODUCTION

The theory of fuzzy functions and its application has increased recently due to the industrial interest of fuzzy control. Moreover, in view of the development of the calculus for fuzzy functions, the investigation of fuzzy differential equations has been initiated and the existence and uniqueness of solutions of fuzzy initial value problem was considered under a Lipschitz condition (Kaleva, 1987; Kaleva, 1990). The basic result such as existence, uniqueness, continuity with respect to initial values and global existence, are proved by developing the needed general comparison principle which helps to understand the intricacies involved in incorporating fuzziness in differential equations (Lakshmikantham,).

Seikkala (1987) proved the existence and uniqueness of the fuzzy solution for the following systems:

$$x^{'}(t) = f(t, x(t)), \qquad x(0) = \tilde{x}_0$$

where f is a continuous mapping from $R^{+} \times R$ into R and $\tilde{x}_0$ is a fuzzy number. Recently, the above concept has been extended to the integro-differential equations by Balasubramaniam and Muralisankar, (2001).

Existence of fixed points in partially ordered sets has been considered resently in (Harjani, 2009; Harjani, 2010; Harjani, 2010). In (Bhaskar, 2006; Dz. Burgic, 2009) some fixed point theorems are proved for a mixed monotone mapping in a metric space endowed with a partial order and the authors apply their results to problems of existence and uniqueness of solutions for some boundary value problems.

The structure of this paper is organized as follows: In Section 2, the basic concept, fuzzy number, Hausdorff metric, generalized H-differentiability and fuzzy ranking are brought. In Section 3, the main section of the paper, some fixed point theorems for the existence and uniqueness of solution to the periodic first-order fuzzy differential equation with boundary value are proved. Finally conclusion is drawn in Section 4.

2 Basic Concept:

A nonempty subset A of R is called convex if and only if $(1-k)x + ky \in A$ for every $x, y \in A$ and $k \in [0,1]$. By $p_k(R)$, we denote the family of all nonempty compact convex subsets of R.

There are various definitions for the concept of fuzzy numbers (Dubois, 1982; Gal, 2000).

Definition 2.1:

A fuzzy number is a function $u : R \to [0,1]$ *satisfying the following properties:*

(i) u is normal, i.e. $\exists x_0 \in R$ with $u(x_0) = 1$,

(ii) u is a convex fuzzy set (i.e. $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\} \forall x, y \in R, \lambda \in [0,1]$),

(iii) u is upper semi-continuous on R,

(iv) $\overline{\{x \in R : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

The set of all fuzzy real numbers is denoted by E. Obviously $R \subset E$. Here $R \subset E$ is understood as

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$R = \{\ \chi_x : \chi$ is usual real number}. For $0 < r \leq 1$, denote $[u]_r = \{x \in R; u(x) \geq r\}$ and $[u]_0 = \overline{\{x \in R; u(x) > 0\}}$. Then it is well-known that for any $r \in [0,1]$, $[u]_r$ is a bounded closed interval. For $u, v \in E$, and $\lambda \in R$, where sum $u+v$ and the product $\lambda.u$ are defined by $[u+v]_r = [u]_r + [v]_r$, $[\lambda.u]_r = \lambda[u]_r$, $\forall r \in [0,1]$, where $[u]_r + [v]_r = \{x + y : x \in [u]_r, y \in [v]_r\}$ means the conventional addition of two intervals (subsets) of R and $\lambda[u]_r = \{\lambda x : x \in [u]_r\}$ means the conventional product between a scalar and a subset of R (Dubois, 1982; Wu, 2001).

Another definition for a fuzzy number is as follows:

Definition 2.2:

An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements:

- $\underline{u}(r)$ is a bounded left-continuous non-decreasing function over $[0,1]$.
- $\overline{u}(r)$ is a bounded left-continuous non-increasing function over $[0,1]$.
- $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$. We recall that for $a < b < c$, $a, b, c \in R$, the triangular fuzzy number $u = (a, b, c)$ determined by a, b, c is given such that $\underline{u}(r) = a + (b - c)r$ and $\overline{u}(r) = c - (c - b)r$ are the endpoints of the r-level sets, for all $r \in [0,1]$. Here $\underline{u}(r) = \overline{u}(r) = b$ and it is denoted by $[u]_1$. For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ we define addition and multiplication by k as

- $\underline{(u+v)}(r) = (\underline{u}(r) + \underline{v}(r))$,
- $\overline{(u+v)}(r) = (\overline{u}(r) + \overline{v}(r))$,
- $\underline{(ku)}(r) = k\underline{u}(r), \overline{(ku)}(r) = k\overline{u}(r), \quad k \geq 0$,
- $\underline{(ku)}(r) = k\overline{u}(r), \overline{(ku)}(r) = k\underline{u}(r), \quad k < 0$.

In this paper, we represent an arbitrary fuzzy number with compact support by a pair of functions $(\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$. Also, we use the Hausdorff distance between fuzzy numbers. This fuzzy number space as shown in [6] can be embedded into Banach space $B = \overline{c}[0,1] \times \overline{c}[0,1]$ where the metric is usually defined as follows: Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded $r-$level sets. Since the $r-$cuts of fuzzy numbers are always closed and bounded, the intervals are written as $u[r] = [\underline{u}(r), \overline{u}(r)]$, for all r. We denote by ω the set of all nonempty compact subsets of R and by ω_c the subsets of ω consisting of nonempty convex compact sets. Recall that

$$\rho(x, A) = \min_{a \in A} \|x - a\|$$

is the distance of a point $x \in R$ from $A \in \omega$ and the *Hausdorff separation* $\rho(A, B)$ of $A, B \in \omega$ is defined as

$$\rho(A, B) = \max_{a \in A} \rho(a, B).$$

Note that the notation is consistent, since $\rho(a, B) = \rho(\{a\}, B)$. Now, ρ is not a metric. In fact, $\rho(A, B) = 0$ if and only if $A \subseteq B.$ The *Hausdorff metric* d_H on ω is defined by

$$d_H(A,B) = \max\{\rho(A,B), \rho(B,A)\}.$$

The metric d_∞ is defined on E as

$$d_\infty(u,v) = \sup\{d_H(u[r],v[r]) : 0 \le r \le 1\},\ u,v \in E.$$

for arbitrary $(u,v) \in \bar{c}[0,1] \times \bar{c}[0,1]$. The following properties are well-known. (Wu, 2001; Gal, 2000).

- $d_\infty(u+w, v+w) = d_\infty(u,v)$, $\forall u,v,w \in E$,
- $d_\infty(k.u, k.v) = |k|\, d_\infty(u,v)$, $\forall k \in R, u,v \in E$,
- $d_\infty(u+v, w+e) \le d_\infty(u,w) + d_\infty(v,e)$, $\forall u,v,w,e \in E$,
- $d_\infty(u,v) = d_\infty(v,u)$, $\forall u,v \in E$.

Theorem 2.1:

- If we define $\bar{0} = \chi_0$, then $\bar{0} \in E$ is a neutral element with respect to addition, i.e. $u + \bar{0} = \bar{0} + u = u$, for all $u \in E$.
- With respect to $\bar{0}$, none of $u \in E \setminus R$, has opposite in E .
- For any $a,b \in R$ with $a,b \ge 0$ or $a,b \le 0$ and any $u \in E$, we have $(a+b).u = a.u + b.u$; however, this relation dose not necessarily hold for any $a,b \in R$, in general.
- For any $\lambda \in R$ and any $u,v \in E$, we have $\lambda.(u+v) = \lambda.u + \lambda.v$.
- For any $\lambda, \mu \in R$ and any $u \in E$, we have $\lambda.(\mu.u) = (\lambda.\mu).u$.(Wu, 2001)

Remark 2.1:

$$d_\infty(u,0) = d_\infty(0,u) = \|u\|$$

Definition 2.3:

Consider $x, y \in E$ *. If there exists* $z \in E$ *such that* $x = y + z$ *, then* z *is called the H-difference of x and y and it is denoted by* $x -_H y$.

In this paper, the sign " $-_H$ " always stands for H-difference and note that $x -_H y \ne x + (-y)$. Let us recall the definition of strongly generalized differentiability introduced in (Bede, 2005).

Lemma 2.1:

Bede, (2005). *Let* $u,v \in E$ *be such that* $u(1) - \underline{u}(0) > 0$, $\bar{u}(0) - u(1) > 0$ *and* $len(v) = (\bar{v}(0) - \underline{v}(0)) \le \min\{u(1) - \underline{u}(0), \bar{u}(0) - u(1)\}$ *. Then the H-difference* $u -_H v$ *exists.*

Definition 2.4:

Let E *be a set of all fuzzy numbers, we say that* $f(x)$ *is a fuzzy function if* $f : E \to E$.

Definition 2.5:

The fuzzy function $f : E \to E$ *is continuous if for arbitrary fixed* $x_0 \in E$ *and* $\varepsilon > 0$ *, exists* $\delta > 0$ *such that*

$$d_\infty(x, x_0) < \delta \quad \Rightarrow \quad d_\infty(f(x), f(x_0)) < \varepsilon$$

Definition 2.6:

For arbitrary fuzzy numbers u *and* v *, define the ranking of u and v by the Hausdorff metric on* E *, i.e.,*

- $u \preceq v$ means that $d_\infty(u, \tilde{0}) \le d_\infty(v, \tilde{0})$
- $u \succeq v$ means that $d_\infty(u, \tilde{0}) \ge d_\infty(v, \tilde{0})$

- $u \approx v$ means that $d_\infty(u,\tilde{0}) = d_\infty(v,\tilde{0})$

Definition 2.7:

Let $f : E \to E$ is a fuzzy function. Then f is non-decreasing fuzzy function if for arbitrary fuzzy numbers x_1 and x_2 we have

$$d_\infty(x_1,\tilde{0}) \le d_\infty(x_2,\tilde{0}) \Rightarrow d_\infty(f(x_1),\tilde{0}) \le d_\infty(f(x_2),\tilde{0})$$

or

$$x_1 \preceq x_2 \Rightarrow f(x_1) \preceq f(x_2)$$

Definition 2.8:

Bede, 2007. *Let $f : (a,b) \to E$ and $x_0 \in (a,b)$. We say that f is strongly generalized differentiable at x_0 (Bede-Gal differentiability), if there exists an element $f'(x_0) \in E$, such that*

(i) for all $h > 0$ sufficiently small, $\exists f(x_0+h) -_H f(x_0)$, $\exists f(x_0) -_H f(x_0-h)$ and the limits(in the metric d_∞)

$$\lim_{h\to 0}\frac{f(x_0+h) -_H f(x_0)}{h} = \lim_{h\to 0}\frac{f(x_0) -_H f(x_0-h)}{h} = f'(x_0)$$

or

(ii) for all $h > 0$ sufficiently small, $\exists f(x_0) -_H f(x_0+h)$, $\exists f(x_0-h) -_H f(x_0)$ and the limits(in the metric d_∞)

$$\lim_{h\to 0}\frac{f(x_0) -_H f(x_0+h)}{-h} = \lim_{h\to 0}\frac{f(x_0-h) -_H f(x_0)}{-h} = f'(x_0)$$

or

(iii) for all $h > 0$ sufficiently small, $\exists f(x_0+h) -_H f(x_0)$, $\exists f(x_0-h) -_H f(x_0)$ and the limits(in the metric d_∞)

$$\lim_{h\to 0}\frac{f(x_0+h) -_H f(x_0)}{h} = \lim_{h\to 0}\frac{f(x_0-h) -_H f(x_0)}{-h} = f'(x_0)$$

or

(iv) for all $h > 0$ sufficiently small, $\exists f(x_0) -_H f(x_0+h)$, $\exists f(x_0) -_H f(x_0-h)$ and the limits(in the metric d_∞)

$$\lim_{h\to 0}\frac{f(x_0) -_H f(x_0+h)}{-h} = \lim_{h\to 0}\frac{f(x_0) -_H f(x_0-h)}{h} = f'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}$ and $\frac{-1}{h}$, respectively)

Proposition 2.1:

Dubois, (1982). *If* $f:(a,b)\to E$ *is a continuous fuzzy valued function then* $g(x)=\int_a^x f(t)d$ *is differentiable with derivative* $g'(x)=f(x)$.

Lemma 2.2:

Bede, 2006. *For* $x_0\in R$, *the fuzzy differential equation* $y' = f(x,y)$, $y(x_0)=y_0\in E$ *where* $f:R\times E\to E$ *is supposed to be continuous, is equivalent to one of the integral equations:*

$$y(x)=y_0+\int_{x_0}^{x} f(t,y(t))dt,\quad \forall x\in[x_0,x_1]$$

or

$$y_0=y(x)+(-1).\int_{x_0}^{x} f(t,y(t))dt,\quad \forall x\in[x_0,x_1]$$

on some interval $(x_0,x_1)\subset R$, depending on the strong differentiability considered, (i) or (ii), respectively.

Here the equivalence between two equations means that any solution of an equation is a solution for the other one, too.

Remark 2.2:

Bede, 2006. *In the case of strongly generalized differentiability, to the fuzzy differential equation* $y' = f(x,y)$ *we may attach two different integral equations, while in the case of differentiability in the sense of the Definition of H-differentiable, we may attach only one. The second integral equation in Lemma (2.2) can be written in the form* $y(x)=y_0 -_H (-1).\int_{x_0}^{x} f(t,y(t))dt$.

Definition 2.9** , (Zimmermann, 1996).* ***(Cartesian Product of Fuzzy Sets):

Let $\tilde{A}_1,\ldots,\tilde{A}_n$ be fuzzy sets in $X_1,\ldots,X_n$. The Cartesian product is then a fuzzy set in the product space $X_1\times\cdots\times X_n$ with the membership function

$$\mu_{\tilde{A}_1\times\cdots\times\tilde{A}_n}(x)=\min_i\{\mu_{\tilde{A}_i}(x_i)\,|\,x=(x_1,\ldots,x_n),x_i\in X_i\}$$

Definition 2.10:

Zimmermann, 1996. *Let* $X,Y\subseteq\Re$ *and*

$$\tilde{A}=\{(x,\mu_{\tilde{A}}(x))\,|\,x\in X\}$$
$$\tilde{B}=\{(y,\mu_{\tilde{B}}(y))\,|\,y\in Y\}$$

are two fuzzy sets. Then

$$\tilde{R}=\{[(x,y),\mu_{\tilde{R}}(x,y)]\,|\,(x,y)\in X\times Y\}$$

is a fuzzy relation on $\tilde{A}$ and $\tilde{B}$ if

$$\mu_{\tilde{R}}(x,y)\le\mu_{\tilde{A}}(x),\quad\forall(x,y)\in X\times Y$$

and

$$\mu_{\tilde{R}}(x,y) \le \mu_{\tilde{B}}(y), \quad \forall (x,y) \in X \times Y$$

Definition 2.11: (Zimmermann, 1996). (Max-min composition):

Let $\tilde{R}_1(x,y)$, $(x,y) \in X \times Y$ and $\tilde{R}_2(y,z)$, $(y,z) \in Y \times Z$ be two fuzzy relations. The max-min composition $\tilde{R}_1$ and $\tilde{R}_2$ is the fuzzy set

$$\tilde{R}_1 \circ \tilde{R}_2 = \{ [(x,z), \max_y \{\min\{\mu_{\tilde{R}_1}(x,y), \mu_{\tilde{R}_2}(y,z)\}\}] \mid x \in X, y \in Y, z \in Z\}$$

$\mu_{\tilde{R}_1 \circ \tilde{R}_2}$ is again the membership function of a fuzzy relation on fuzzy sets.

3 Main Results:

In the this section, we are going to study the uniqueness and existence of solutions to the periodic first-order fuzzy differential equation with boundary value.

Consider the periodic first-order fuzzy differential equation with boundary value

$$\begin{cases} y'(t) = f(t, y(t)), & t \in I = [0,T] \\ y(0) = y(T) \in E \end{cases} \tag{1}$$

where $T > 0$, $f : I \times E \to E$ is continuous fuzzy function such that $f(t,y)$ is non-decreasing in the second variable.

Definition 3.1:

A continuous fuzzy valued function $\alpha(t)$ is called minimal fuzzy solution of (1), if

$$\begin{cases} d_\infty(\alpha'(t), \tilde{0}) \le d_\infty(f(t,\alpha(t)), \tilde{0}) \\ d_\infty(\alpha(0), \tilde{0}) \le d_\infty(\alpha(T), \tilde{0}) \end{cases} \tag{2}$$

Definition 3.2:

A continuous fuzzy valued function $\alpha(t)$ is called maximal fuzzy solution of (1), if

$$\begin{cases} d_\infty(\alpha'(t), \tilde{0}) \ge d_\infty(f(t,\alpha(t)), \tilde{0}) \\ d_\infty(\alpha(0), \tilde{0}) \ge d_\infty(\alpha(T), \tilde{0}) \end{cases} \tag{3}$$

Theorem 3.1:

Let $f : E \to E$ is a continuous and non-decreasing fuzzy function and (E, d_∞) is a complete fuzzy metric space. Also, exists $k \in [0,1)$, such that

$$d_\infty(f(t), f(z)) \le k d_\infty(t,z), \qquad \forall\, t \preceq z \tag{4}$$

If exists $t_0 \in E$, such that $d_\infty(t_0, \tilde{0}) \le d_\infty(f(t_0), \tilde{0})$ then, the fuzzy function f has a fixed point.

Proof:

If $d_\infty(t_0, \tilde{0}) = d_\infty(f(t_0), \tilde{0})$, then the proof is finished. so suppose $d_\infty(t_0, \tilde{0}) < d_\infty(f(t_0), \tilde{0})$ and since f is non-decreasing, we have

$$t_0 \prec f(t_0) \quad \Leftrightarrow d_\infty(t_0,\tilde{0}) < d_\infty(f(t_0),\tilde{0})$$
$$f(t_0) \prec f(f(t_0)) \quad \Leftrightarrow d_\infty(f(t_0),\tilde{0}) < d_\infty(f(f(t_0)),\tilde{0})$$

then

$$d_\infty(t_0,\tilde{0}) < d_\infty(f(f(t_0)),\tilde{0}) = d_\infty(f^2(t_0),\tilde{0})$$

If we continue this procedure, we can have

$$d_\infty(t_0,\tilde{0}) < d_\infty(f^{n+1}(t_0),\tilde{0})$$

Now, we prove Eq. (5) by using mathematical induction,

$$d_\infty(f^{n+1}(t_0), f^n(t_0)) \le k^n d_\infty(f(t_0),t_0) \tag{5}$$

First, we suppose $n=1$. If we replace $z = f(t_0)$ and $t = t_0$ in Eq. (7), then

$$d_\infty(f(t_0), f^2(t_0)) \le k d_\infty(t_0, f(t_0))$$

Suppose for $n = m$, Eq. (5) is true. So, for $n = m+1$ we have

$$\begin{aligned} d_\infty(f^{m+2}(t_0), f^{m+1}(t_0)) &= d_\infty(f^{m+1}(f(t_0)), f^m(f(t_0))) \\ &\le k^m d_\infty(f^2(t_0), f(t_0)) \\ &\le k^{m+1} d_\infty(t_0, f(t_0)) \end{aligned}$$

therefore, Eq. (5) holds. If $p > n$, then

$$\begin{aligned} d_\infty(f^p(t_0), f^n(t_0)) &= d_\infty(f^p(t_0) + f^{p-1}(t_0), f^{p-1}(t_0) + f^n(t_0)) \\ &\le d_\infty(f^p(t_0), f^{p-1}(t_0)) + d_\infty(f^{p-1}(t_0), f^n(t_0)) \\ &\le d_\infty(f^p(t_0), f^{p-1}(t_0)) + d_\infty(f^{p-1}(t_0) + f^{p-2}(t_0), f^{p-2}(t_0) + f^n(t_0)) \\ &\le d_\infty(f^p(t_0), f^{p-1}(t_0)) + d_\infty(f^{p-1}(t_0), f^{p-2}(t_0)) \\ &+ d_\infty(f^{p-2}(t_0), f^n(t_0)) \\ &\vdots \\ &\le d_\infty(f^p(t_0), f^{p-1}(t_0)) + d_\infty(f^{p-1}(t_0), f^{p-2}(t_0)) \\ &+ \cdots + d_\infty(f^{n+1}(t_0), f^n(t_0)) \\ &\le k^{p-1} d_\infty(f(t_0),t_0) + k^{p-2} d_\infty(f(t_0),t_0) + \cdots + k^n d_\infty(f(t_0),t_0) \\ &= d_\infty(f(t_0),t_0)\left(k^{p-1} + k^{p-2} + \cdots + k^n\right) \\ &= \frac{k^n - k^p}{1-k} d_\infty(f(t_0),t_0) \\ &\le \frac{k^n}{1-k} d_\infty(f(t_0),t_0) \end{aligned}$$

thus, $\{f^n(t_0)\}_{n \in N}$ is a Cauchy sequence. Since (E, d_∞) is a complete fuzzy metric space, therefore exists $z \in E$, such that

$$d_\infty(f^n(t_0), z) \to 0 \tag{6}$$

for $n \to \infty$.

Now, we represent $z \in E$ is a fixed point for f i.e., $d_\infty(f(z),\tilde{0}) = d_\infty(z,\tilde{0})$. Using Eq. (6), for $n \to \infty$ we can also assume

$$\begin{aligned} d_\infty(f(z),z) &\approx d_\infty(f(z), f^n(t_0)) \\ &\le k d_\infty(z, f^{n-1}(t_0)) \\ &< d_\infty(z, f^{n-1}(t_0)) \to 0 \end{aligned}$$

for $k < 1$. Hence,

$$d_\infty(f(z),z) \to 0$$

so, z is a fixed point for f.

Theorem 3.2:

Let $f : E \to E$ is a continuous and non-decreasing fuzzy function and (E, d_∞) is a complete fuzzy metric space. Also, if $\{t_n\}$ is a nondecreasing sequence with $t_n \to t$ in E, then for all $n \in N$

$$d_\infty(t_n,\tilde{0}) \le d_\infty(t,\tilde{0})$$

Also, exists $k \in [0,1)$, such that

$$d_\infty(f(t), f(z)) \le k d_\infty(t,z), \qquad \forall\, t \preceq z \tag{7}$$

If exists $t_0 \in E$, such that $d_\infty(t_0,\tilde{0}) \le d_\infty(f(t_0),\tilde{0})$ then, the fuzzy function f has a fixed point.

Proof:

Similar to the proof of Theorem (3.1), we can derive $\{f^n(t_0)\}_{n \in N}$ is a Cauchy sequence and convergent to $z \in E$. It is sufficient to show that z is a fixed point for f.

Suppose $\varepsilon > 0$ and exists $n_1 \in N$ such that for $n \ge n_1$,

$$d_\infty(f^n(t_0), z) \le \frac{\varepsilon}{2}$$

Since $d_\infty(f^n(t_0),\tilde{0}) \le d_\infty(z,\tilde{0})$ for all $n \in N$, so,

$$\begin{aligned} d_\infty(f(z),z) &\le d_\infty(f(z), f^{n+1}(t_0)) + d_\infty(f^{n+1}(t_0), z) \\ &= d_\infty(f(z), f(f^n(t_0))) + d_\infty(f^{n+1}(t_0), z) \\ &\le k d_\infty(z, f^n(t_0)) + d_\infty(f^{n+1}(t_0), z) \\ &\le d_\infty(z, f^n(t_0)) + d_\infty(f^{n+1}(t_0), z) \\ &< \varepsilon \end{aligned}$$

therefore, z is a fixed point for f.

Theorem 3.3:

The fixed point z for f is uniqueness.

Proof:

Suppose that f has another fixed point u, i.e., $d_\infty(f(u),u) \to 0$. We must prove

$$d_\infty(u,z)=0$$

To do this, we have

$$\begin{aligned} d_\infty(u,z) &= d_\infty(z+f(u), f(u)+u) \\ &\leq d_\infty(z, f(u)) + d_\infty(f(u),u) \\ &= d_\infty(f(u),z) \\ &= d_\infty(f(u)+f(z), f(z)+z) \\ &\leq d_\infty(f(u), f(z)) + d_\infty(z, f(z)) \\ &\leq k d_\infty(u,z) \\ &< d_\infty(u,z) \end{aligned}$$

for $k<1$. So,

$$d_\infty(u,z) < d_\infty(u,z)$$

that it is not true. Then, $d_\infty(u,z)=0$ and the fixed point z for f is uniqueness.

Theorem 3.4:
Let exist $\lambda,\mu>0$ *and* $\lambda>\mu$*, such that for all* $x,y\in E$

$$0 \leq d_\infty(f(t,y)+\lambda y, f(t,x)+\lambda x) \leq \mu d_\infty(y,x), \quad y \succeq x \tag{8}$$

Then, the existence of minimal solution to the periodic first-order fuzzy differential equation with boundary value (1) guarantees the existence of the maximal solution of it.

Proof:
We can rewrite Eq. (1) as follows:

$$\begin{cases} y'(t)+\lambda y(t) = f(t,y(t))+\lambda y(t), & t\in I=[0,T] \\ y(0)=y(T)\in E \end{cases} \tag{9}$$

Then, Eq. (1) is written, equivalently, as the integral equation:

$$y(t)=\int_0^T G(t,s)\big(f(s,y(s))+\lambda y(s)\big)ds$$

where

$$G(t,s)=\begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0\leq s<t\leq T \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0\leq t<s\leq T \end{cases}$$

is the Green function.

Let the operator $A: C(I,E)\to C(I,E)$ is defined for all $t\in I$ as follows:

$$[Ay](t) : \int_0^T G(t,s)\big(f(s,y(s)) + \lambda y(s)\big)ds \tag{10}$$

If $y \in C(I,E)$ is a fixed point of A, then $y \in C(I,E)$ is a solution of (1). So, we need only present that $y \in C(I,E)$ is a fixed point of A. To do this, we present that the assumptions of theorem (3.2) and (3.3) satisfy.

Consider the complete metric space $(C(I,E), d_\infty)$ and the partial order on $C(I,E)$ as follows:

$$x \preceq y \quad \Leftrightarrow \quad x(t) \preceq y(t), \quad \forall t \in I$$

Let $\{x_n\} \subseteq C(I,E)$ is a non-decreasing sequence and is convergence to x in $C(I,E)$, then

$$x_1(t) \preceq x_2(t) \preceq \cdots \preceq x_n(t) \preceq \cdots$$

so, we can say

$$x_n(t) \preceq x(t), \qquad \forall n \in N$$

thus,

$$x_n \preceq x, \qquad \forall n \in N$$

and x is an upper bound of $\{x_n\}$.

For $u \succeq v$, we have

$$f(t,u) + \lambda u \succeq f(t,v) + \lambda v$$

and since

$$G(t,s) > 0, \quad (t,s) \in I \times I$$

then, for $t \in I$

$$\begin{aligned}[Au](t) \quad &\approx \int_0^T G(t,s)\big(f(s,u(s)) + \lambda u(s)\big)ds \\ &\succeq \int_0^T G(t,s)\big(f(s,v(s)) + \lambda v(s)\big)ds \\ &\approx [Av](t)\end{aligned}$$

therefore, A ia a non-decreasing operator. Also,

$$\begin{aligned}
d_\infty(Au,Av) &= d_\infty\left(\int_0^T\big(f(s,u(s))+\lambda u(s)\big)ds, \int_0^T G(t,s)\big(f(s,v(s))+\lambda v(s)\big)ds\right)\\
&\le \int_0^T G(t,s) d_\infty(f(s,u(s))+\lambda u(s), f(s,v(s))+\lambda v(s))ds\\
&\le \int_0^T G(t,s)\mu d_\infty(u,v)ds\\
&\le \mu d_\infty(u,v)\int_0^T G(t,s)ds\\
&= \mu d_\infty(u,v)\frac{1}{e^{\lambda T-1}}\left(\left.\frac{1}{\lambda}e^{\lambda(T+s-t)}\right]_0^t + \left.\frac{1}{\lambda}e^{\lambda(s-t)}\right]_t^T\right)\\
&= \mu d_\infty(u,v)\frac{1}{\lambda e^{\lambda T-1}}\left(e^{\lambda T-1}\right)\\
&= \frac{\mu}{\lambda}d_\infty(u,v)
\end{aligned}$$

since $k=\dfrac{\mu}{\lambda}<1$ then the assumption of theorem (3.2) is satisfied.

Now, we suppose that exists $\alpha \in C(I,E)$ such that $\alpha \le A(\alpha)$ then, A has a fixed point in $C(I,E)$ We have

$$\alpha'(t)+\lambda\alpha(t) \preceq f(t,\alpha(t))+\lambda\alpha(t), \quad t\in I$$

so,

$$e^{\lambda t}(\alpha'(t)+\lambda\alpha(t)) \preceq e^{\lambda t}(f(t,\alpha(t))+\lambda\alpha(t)), \quad t\in I$$

If $\alpha(t)$ is $i-$differentiable, then,

$$(e^{\lambda t}\alpha(t))' \preceq e^{\lambda t}(f(t,\alpha(t))+\lambda\alpha(t)), \quad t\in I$$

and so,

$$e^{\lambda t}\alpha(t) \preceq \alpha(0)+\int_0^t e^{\lambda s}(f(s,\alpha(s))+\lambda\alpha(s))ds, \quad t\in I \tag{11}$$

Thus, we derive

$$e^{\lambda t}\alpha(t) \preceq e^{\lambda T}\alpha(T) \preceq \alpha(0)+\int_0^t e^{\lambda s}(f(s,\alpha(s))+\lambda\alpha(s))ds$$

then,

$$\alpha(0) \preceq \int_0^t \frac{e^{\lambda s}}{e^{\lambda t}-1}(f(s,\alpha(s))+\lambda\alpha(s))ds \tag{12}$$

Using Eqs. (11) and (12), for all $t\in I$, we have

$$\begin{aligned}
e^{\lambda t}\alpha(t) \quad &\circ \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T}-1}(f(s,\alpha(s))+\lambda\alpha(s))ds\\
&+\int_t^T \frac{e^{\lambda s}}{e^{\lambda T}-1}(f(s,\alpha(s))+\lambda\alpha(s))ds
\end{aligned}$$

and finally, we can derive

$$\alpha(t) \preceq \int_0^T G(t,s)(f(s,\alpha(s)) + \lambda\alpha(s))ds \square [A\alpha](t)$$

So, by using theorem (3.3), the fixed point is uniqueness.

4 Conclusion:

In this work, we studied the existence and uniqueness of solution to the periodic first-order fuzzy differential equation with boundary value using fixed point theorem. For this idea, we defined the Minimal and maximal solutions of it and then proved some theorems in detail.

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