Australian Journal of Basic and Applied Sciences, 6(10): 48-52, 2012
ISSN 1991-8178

The Computation of the Commutativity Degree for Dihedral Groups in Terms of Centralizers

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Abstract: The commutativity degree of finite groups is computed by finding the number of conjugacy classes of G. Also, finding the centralizers of a finite group can be applied to obtain the commutativity degree of the group. In this paper we construct some upper bounds for the commutativity degree in terms of centralizers for the dihedral group D_n, where the dihedral group acts on itself by conjugation.

Key words: Commutativity degree; dihedral group; group action.

INTRODUCTION

Throughout this paper G denotes a finite group. The commutativity degree is the probability that a random element x commute with another random element y in G. This probability is denoted by $P(G)$ and defined as follows :

$$P(G) = \frac{|\{x, y) \in G \times G : xy = yx\}|}{|G|^2}.$$

This concept was introduced in 1968 by Erdos and Turan, where they worked on symmetric groups. Few years later Gustafson (1973) and MacHale (1974) proved that this probability is less than or equal to 5/8. The concept of commutativity degree has been extended by using other group structures and more results were obtained associated with this concept. For instance, some researches have been successfully done on commutator of subgroups and constructed values for upper and lower bounds (see Das and Nath, 2010, for more details).

In this paper, we determine the commutativity degree in terms of centralizers for the dihedral groups.

The followings are some definitions that are used in this research.

Definition 1.1:

(Gallian, J. A. 2002) The dihedral group of order $2n$ is a group generated by two elements a and b. The group is known as the group of symmetries of a regular n-gon. The presentation of this group can be expressed as follows:

$$D_n = \langle a, b: a^2 = b^2 = (ab)^2 = e, bab = a^{-1} \rangle.$$

Definition 1.2:

(Rotman, 2002) Let G be any finite group and X be a set. G acts on X if there is a function $G \times X \rightarrow X$, such that.

(1) $(gh)x = g(hx)$, $\forall\, g, h \in G, x \in X$.
(2) $1_G\, x = x$, $\forall\, x \in G$.

Definition 1.3:

(Goodman, 2003) Let G act on a set S, and $x \in S$. If G acts on itself by conjugation, the orbit $O(x)$ is defined as follows:

$\{y \in G : y = axa^{-1}$ for some $a \in G\}$.

In this case $O(x)$ is called the conjugacy classes of x in G. Throughout this paper, we use K as a notation for the number of conjugacy classes in G.

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Definition 1.6:

(Goodman, 2003) Let G act on a set S, and $x \in S$. The centralizer of x under the action of G on S is a subgroup

$$\mathrm{Cent}_G(x) = \{ g \in G : gxg^{-1} = x \}.$$

This subgroup is often called the isotropy group of x in G and it is known as the centralizer of x in G.

In 1997, Achar studied the properties of the size of conjugacy classes in case that G is a semigroup of endomorphism associated on action $\phi : G \rightarrow S_G$. The action is stated as follows:

$$g * x = \phi(g)xg^{-1} .$$

This action is known as ϕ-conjugate. The ϕ-conjugacy class is the orbit of x under the action and denoted by C_x^{ϕ} . The stabilizer of x under the action is named by the ϕ-centralizer of x and is denoted as Z_x^{ϕ}.

The next section focuses on the commutativity degree of the dihedral group in terms of centralizers.

Preliminaries:

In this section, we provide some definitions that are related to commutativity degree. Pournaki and Sobhani (2008) studied the probability that the commutator of two group elements is equal to a given element in a group G. The probability is stated as follows:

$$P_g(G) = \frac{|\{(x,y) \in G \times G \,|[x,y] = g\}|}{|G|^2}.$$

This probability was also studied by Alghamdi and Russo (2011) where some results were obtained.

In 2010, Das and Nath studied the probability that the commutator of two subgroups elements is equal to an element of a group G and the probability is defined as:

$$P(H,K) = \frac{|(h,k) \in H \times K : hk\,h^{-1}k^{-1} = g\}|}{|H|\,|K|}.$$

A new concept for the commutativity degree was introduced by Sherman (1975) which he introduced the probability of an automorphism of a finite group which fixes an arbitrary element. This concept is defined as follows:

$$P_G(X) = \frac{|\{g,x)|gx = x \;\; \forall\, g \in G, x \in X\}|}{|X||G|}.$$

Later in 2011, further explorations on Sherman's definition were done by Moghaddam et al. (2011), where they introduced a new probability which is called the probability of an automorphism fixes a subgroup element of a finite group. This probability is stated as follows:

$$P_{AG}(H,G) = \frac{|\{(\alpha,h)|\, h^{\alpha}\,, h \in H, \alpha \in A_G\}|}{|H||G|}.$$

In case that $H = G$, then $P_{A_G}(G,G) = P_{A_G}(G)$. Among other results, some upper and lower bounds were obtained (Moghaddam et al., 2011). In addition, some researchers tend to study the commutativity degree in case that the group is affected by some actions, one of these studies is mentioned in previous section practically the work that was done by Achar (1997). Another type of commutativity degree of G, denoted by Prcom_{ϕ} , is defined as follows:

$$\mathrm{Prcom}_{\phi} = \frac{K_{\phi}}{|G|},$$

where K_{ϕ} is the number of ϕ-conjugacy classes. The conjugacy classes' structure hence is produced by endomorphisms.

The main objective of this paper is to find the centralizers which are needed in the computation of the commutativity degree in term of centralizers. This work is discussed in the next section.

Numerical Computation:

In this section, the focus is given on the dihedral group D_n, where we are going to find the centralizers of this group. As a consequence, some upper bounds for the commutativity degree of the dihedral group in terms of centralizers are provided and some primary results based on this work are also found.

The notation Prcom_ϕ is used for the commutativity degree of dihedral groups in terms of the centralizers.

The function ϕ is defined as follows:

Let ϕ be a homomorphism from D_n to D_n. The dihedral group D_n acts on itself by conjugation so that ϕ-conjugacy class of D_n is given as: $C_x^\phi = \{\phi(g)xg^{-1} : g \in G\}$. and the ϕ-centralizer is given as $Z_x^\phi = \{\phi(g)x\, g^{-1} = x : g\, \in G\}$. In order to find a formula for Prcom_ϕ the number of the centralizers of elements of D_n should be first found. The computation is done up to order 100 and the results obtained is described in Table 1.

Table 1: The number of ϕ-conjugacy classes and ϕ-centralizers for D_n

$\lvert D_n \rvert$	*The number of ϕ-conjugacy classes, C_x^ϕ*	*r of ϕ-centralizers,Z_x^ϕ*
6	*3*	*5*
8	*5*	*4*
10	*4*	*7*
12	*6*	*5*
14	*5*	*9*
16	*7*	*6*
18	*6*	*11*
20	*8*	*7*
22	*7*	*13*
24	*9*	*8*
26	*8*	*15*
40	*13*	*12*
80	*23*	*22*
100	*28*	*27*

From Table 1, we get the result for commutativity degree in terms of ϕ-centralizers given in the following.

Proposition 3.1:

Let D_n be the dihedral group of order n. If n is a positive integer, then

$$Z(D_n) = \begin{cases} \{1, a^{\frac{n}{2}}\}, & \textit{if n is even.} \\ \{1\}, & \textit{if n is odd.} \end{cases}$$

Proof:

Suppose that $a \in Z(G)$. Since every element in $Z(G)$ commute with other elements in *G*, so $a^j b$ commute, if *j* = 0 or *2j* = *n*. In the case that *n* is even, $j = \frac{n}{2}$. Thus, $Z(G) = \{1, a^{\frac{n}{2}}\}$. On the other hand, when *n* is odd, *j* = 0, thus $Z(D_n) = \{1\}$. ■

Corollary 3.2:

$C_x^\phi > Z_x^\phi$ *if n is even, and* $C_x^\phi < Z_x^\phi$ *if n is odd.*

Proof:

Using Proposition 3.1, it can be concluded that the number of conjugacy classes is greater than the number of the centralizers when the center is of order 2. Whilst, the number of the conjugacy classes is lower than the number of the centralizers when the center has order 1. ■

Theorem 3.1:

Let G be a 2-generator group of order 2n, and let n be a positive integer. If G acts on itself by conjugation, then the commutativity degree of dihedral groups D_n in terms of centralizers is given as follows:

$$\text{Prcom}_\phi = \begin{cases} \dfrac{|\text{Cent}_G(x)| + |Z(G)|}{2|D_n|}, & \textit{if n is even,} \\ \dfrac{|\text{Cent}_G(x)| + 1}{|D_n|}, & \textit{if n is odd,} \end{cases}$$

where *n* is a positive integer.

Proof:

We first show in the case n is even. The isotropy groups in G are $\{D_n, H \lhd G,$ and $\frac{n}{2}$ isotropy groups of order 4$\}$. Using Corollary 3.2, we find that $K(G) = |\mathrm{Cent}_G(x)| + 1$. Thus, $\mathrm{Prcom}_\phi = \frac{|\mathrm{Cent}_G(x)|+1}{|D_n|}$.

Next, we prove in the case n is odd. The number of isotropy groups in G under conjugation is: $\{D_n, H \lhd G$ and n isotropy groups of order 2$\}$. Using Corollary 3.2, we have

$$\mathrm{Prcom}_\phi = \frac{|\mathrm{Cent}_G(x)| + 1}{2|D_n|}. \quad \blacksquare$$

The following are some propositions that can be concluded from our results.

Proposition 3.2:

Let G be a 2-generater group of order 2p, where p a prime number. If G acts on a set S by conjugation, then $\mathrm{P(G)} = \frac{\mathrm{p}+3}{4\mathrm{p}}$.

Proof:

We first find the number of conjugacy classes of G. Let p be a prime. The identity element is one of the conjugacy classes in G. However, there are n elements in the form $[b] = \{b\,a^i : 0 \le i \le p\}$, thus the number of conjugacy classes is one. On the other hand, the number of elements in the form $[a^i] = \{a^i, a^{p-i}\}$ are two, therefore the number of conjugacy classes are $\frac{p-1}{2}$. All elements are included in one of the classes listed above. Therefore, the numbers of conjugacy classes in G are $K(G) = 1 + 1 + \frac{p-1}{2}$. Thus, $K(G) = \frac{p+3}{2}$. Dividing both sides by $|G|$ gives $\frac{K(G)}{|G|} = \frac{\frac{P+3}{2}}{|G|}$. It follows that $P(G) = \frac{p+3}{4p}$. $\blacksquare$

Proposition 3.3:

Let G be a 2-generater group of order 2p, where p is a prime number. If H is a cyclic subgroup of G and |G/H| = p, then $\mathrm{P(G)} = \frac{\mathrm{p}+3}{4\mathrm{p}}$.

Proof:

Suppose that G is a 2-generater group of order $2p$. Let H be a cyclic subgroup of G. Assume $H^i = e$, thus in case of $i = n$, $H \lhd G$. On the other hand, since p is a prime number, the center $Z(G)$ is the identity element, so that the cyclic subgroups of G are of order 2. Thus, $i = 2$ and $|H| = 2$. However, using Proposition 3.2, the number of conjugacy classes in G is: $K(G) = \frac{p+3}{2}$. Dividing both sides by $|G|$ gives $\frac{K(G)}{|G|} = \frac{P+3}{2|G|}$. Since $|G/H| = p$, $|G| = p\,|H|$. It follows that $P(G) = \frac{p+3}{4p|H|}$. Since $|H| = 2$, hence the result follows. $\blacksquare$

Proposition 3.3:

Let D_{pq} be a finite dihedral group of order 2pq, where p, q are prime numbers and p ≠ q. If D_{pq} acts on itself by conjugation, then $\mathrm{P}(\mathrm{D_{pq}}) = \frac{\mathrm{p+q+1}}{|\mathrm{G}|}$.

Proof:

Suppose that p, q are two prime numbers where $p \neq q$, and let D_{pq} be a 2-generater group of order $2pq$. Then, the number of centralizers in D_{pq} is equal to $p + q$. Therefore, using Corollary 3.2, we find that $P(D_{pq}) = \frac{p+q+1}{|G|}$. $\blacksquare$

RESULTS AND CONCLUSIONS

In this paper, a formula for the commutativity degree in terms of centralizers for dihedral groups has been found. Moreover, some upper bounds for the commutativity degree in terms of centralizers for the dihedral groups have been constructed. Besides, some results that are related to the commutativity degree are provided.

ACKNOWLEDGEMENT

The first author would like to acknowledge Universiti Teknologi Malaysia, for her International Doctoral Followship (IDF) for Semester I 2011/2012.

REFERENCES

Achar, P.N., 1997. Generalized conjugacy classes. Technical Report. Rose Hulman Mathematical Sciences Series no., 9701.

Alghamdi, A.M. and F.G. Russo, 2011. A generalized of the probability that commutator of two group elements is equal to a given element. Bulletin of the Iranian Mathematical Society (in press).

Das, A.K. and R.K. Nath, 2010. On generalized relative commutativity degree of a finite group. International Electronic Journal of Algebra, 7: 140-151.

Erdos, P. and P. Turan, 1968. On some problems of a statistical group theory. IV, Acta Math. Acad Sci. Hungaricae, 19: 413-435.

Gallian, J.A., 2002. Contemporary Abstract Algebra. 5th Edition. Boston New York: Houghton Mifflin Company.

Goodman, F.M., 2003. Algebra abstract and concrete. Streesing Symmetry. 2nd Edition. New Jersey 07458: Pearson Education,Inc. Upper Saddle River.

Gustafson, W.H., 1973. What is the probability that two group elements commute? The American Mathematical Monthly, 80(9): 1031-1034.

MacHale, D., 1974. How commutative can a non-commutative group be? The Mathematical Gazette, 58: 199-202.

Moghaddam, M.R.R., F. Saeedi and E. Khamseh, 2011. The probability of an automorphism fixing a subgroup element of a finite group. Asian-European Journal of Mathematics. 4 (2): 301-308.

Pournaki, M. R. and Sobhani, R. (2008). Probability that commutator of two group elements is equal to a given element. J. Pure Appl. Algebra, 212: 727-734.

Rotman, J.J., 2002. Advanced Modren Algebra. New Jersey 07458: Pearson Education,Inc. Upper Saddle River.

Sherman, G.J., 1975. What is the probability an automorphsim fixes a group element? The American Mathematical Monthly, 82(3): 261-264.