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# A Four Point Block Integration Method for the Solutions of IVP in ODE

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**Abstract:** A one step block integration method for initial value problems of general second order ordinary differential equations which combine the Runge-Kutta type one step procedure and the Adam's type multistep procedure is proposed in this paper. Convergence of this sixth order method is established by the consistency and zero stability properties. The method is also characterized by the region of absolute stability. Comparison with existing methods obtained with step number k>1 shows that the new method is comparatively accurate.

**Key words:** Convergence, Zero stability, hybrid, offgrid point, implicit, block method.

## INTRODUCTION

The method proposed in this paper is a one step implicit hybrid block method which can be represented as a matrix difference equation of the form:

$$h^{\tau} AY_m = h^{\tau}\mathbf{y}_m + h^{\mu-\tau}\left[CF\left(y_m\right)+DF\left(Y_m\right)\right] \tag{1}$$

Where A, B, C, D, are constant coefficient matrices, $h^{\tau} y_m = (y_{n-\frac{1}{5}}, y_{n-\frac{2}{5}}, y_{n-\frac{3}{5}}, y_{n-\frac{4}{5}}, y_n, h^{\tau} y'_{n-\frac{1}{5}}, h^{\tau} y'_{n-\frac{2}{5}}, h^{\tau} y'_{n-\frac{3}{5}}, h^{\tau} y'_{n-\frac{4}{5}}, h^{\tau} y'_n)^T$; $h^{\tau}Y_m = (y_{n+\frac{1}{5}}, y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, y_{n+\frac{4}{5}}, y_{n+1}, h^{\tau} y'_{n+\frac{1}{5}}, h^{\tau} y'_{n+\frac{2}{5}}, h^{\tau} y'_{n+\frac{3}{5}}, h^{\tau} y'_{n+\frac{4}{5}}, h^{\tau} y'_{n+1})^T$; $F\left(Y_m\right) = (f_{n-1}, f_{n-\frac{4}{5}}, f_{n-\frac{3}{5}}, f_{n-\frac{2}{5}}, f_{n-\frac{1}{5}}, f_{n+\frac{1}{5}}, f_{n+\frac{2}{5}}, f_{n+\frac{3}{5}}, f_{n+\frac{4}{5}}, f_{n+1})^T$; $F\left(y_m\right) = (f_{n-\frac{1}{5}}, f_{n-\frac{2}{5}}, f_{n-\frac{3}{5}}, f_{n-\frac{4}{5}}, f_{n+1}, f_{n+\frac{4}{5}}, f_{n+\frac{3}{5}}, f_{n+\frac{2}{5}}, f_{n+\frac{1}{5}}, f_n)^T$; $h$ is a fixed step size defined later in Section 2; $\tau$ is the degree of the derivative of (1) and $\mu$ is the order of the ordinary differential equation to be integrated.

Note that apart from being accurate schemes on their own, the block solution obtained from this method can also serve as predictors elsewhere.

The method is proposed for the direct solution of the class of ordinary differential equations of the form:

$$\begin{cases} y'' = f\left(x, y, y'\right) \\ \quad y\left(a\right) = \eta_o \\ \quad y'\left(a\right) = \eta_1 \end{cases} \tag{2}$$

in the interval, $\left[a,\ b\right]$, a, b $\in$ R where $f$ is continuously differentiable. This class of equations usually arises from the mathematical modeling of physical phenomena; for example in dynamic and mechanical systems. The existence and uniqueness of the solution of (2) had been established (see Wend, 1969).

The method (1) is obtained by combining implicit one-step hybrid methods of the form:

$$y\left(t\right) = \sum_{j'=3,4} \alpha_{\nu_{j'}}\left(t\right) y_{n+\nu_{j'}} + h^2\left[\sum_{j=0}^{1} \beta_j\left(t\right) f_{n+j} + \sum_{j'=1}^{4} \beta_{\nu_{j'}}\left(t\right) f_{n+\nu_{j'}}\right], \tag{3}$$

evaluated at selected grid and offgrid points respectively, into a block. In (3), $\alpha_{\nu_{j'}}, \beta_{\nu_{j'}}$ and $\beta_j$ are continuous coefficients of the scaling factor $t \in R$; $y_{n+j} = y\left(x_{n+j}\right)$ is the numerical approximation of the exact solution at the grid point $x_{n+j} = x_n + jh$ and $f_{n+j} = f\left(x_{n+j}, y_{n+j}, y'_{n+j}\right)$. The subscripts $\nu_{j'} \notin \left\{0,1\right\}$ are rational numbers representing the $j^{th}$ offgrid point.

This method combines the advantages presented by the Runge-Kutta procedure of evaluating functions at offgrid points and the accuracy and efficiency of linear multi-step procedure for a given step number compared to the Runge- Kutta methods. Hence, our method is classified as a hybrid method. A unique feature of our approach, is that interpolation is done only at the offgrid points unlike in existing methods available in the literatures; for example in (Kayode, 2011; Awoyemi and Idowu, 2005 and Badmus and Yahaya, 2009)

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In the next section the procedure of derivation of the method is outlined, in section three the implementation procedure is discussed. Analysis of fundamental properties that characterize the new method is discussed in section 4. In section five, the new method is experimented on some sample problems and comparison is made, (in terms of the step method, order of accuracy and the absolute error), with results obtained from existing methods.

***Derivation Of The Method:***

Let $k = 1$, then incorporated four offgrid points, $\nu_{j'}, j'=1,\ldots,4$, between the grid points $x_n$ and $x_{n+1}$. The exact solution of (2) is then approximated in the range $x_n \le x \le x_{n+1}$ with fixed step size $h$, given by $x_{n+1}-x_n$ using the power series polynomial of the form:

$$P(x)=\sum_{j=0}^{m} a_j x^j \,, \tag{4}$$

as the basis polynomial.

Now, interpolate the basis polynomial at two carefully selected offgrid points namely; $x_{n+\nu_3}$ and $x_{n+\nu_4}$ respectively, in a Stormer-Cowell style, (see Fatunla, 1988). Furthermore, collocate the differential system at all the grid and offgrid points, namely; $x_n, x_{n+\nu_{j'}}$ and $x_{n+1}$, respectively. This way, we obtained a system of eight equations, each with degree at most seven viz:

$$\begin{aligned} &P\left(x_{n+\nu_3}\right)=y_{n+\nu_3} \\ &P\left(x_{n+\nu_4}\right)=y_{n+\nu_4} \\ &P''\left(x_{n+j}\right)=f_{n+j}, \quad j=0,\nu_{j'},1 \end{aligned} \tag{5}$$

Solving (5) by Gauss elimination method values for the unknown parameters $a_j, j=0,\nu_{j'},1$, in the polynomials are obtained and substituted into (4). In the range $x_n \le x \le x_{n+1}$, use the scaling factor

$$t=\frac{x-x_{n+\nu_4}}{h} \tag{6}$$

in (4) after substituting the (symbol), to obtain the continuous one step implicit hybrid formula of the form (3).

In particular choose $\nu_1=\frac{1}{5}, \nu_2=\frac{2}{5}, \nu_3=\frac{3}{5}$ and $\nu_4=\frac{4}{5}$. Note that the choice of this offgrid points determines the zero stability or the otherwise of the resultant schemes. Hence, the coefficients $\alpha_{\nu_3}, \alpha_{\nu_4}, \beta_j$ and $\beta_{\nu_{j'}}$ are obtained as follows:

$$\begin{aligned} &\alpha_{\frac{3}{5}}(t)=-5t \\ &\alpha_{\frac{4}{5}}(t)=5t+1 \\ &\beta_0(t)=\frac{-h^2}{10080}\left(6250t^7+875t^6+2625t^5-875t^4-420t^3+8t\right) \\ &\beta_{\frac{1}{5}}(t)=\frac{-h^2}{50400}\left(156250t^7+262500t^6+91875t^5-26250t^4-14000t^3+277t\right) \\ &\beta_{\frac{2}{5}}(t)=\frac{-h^2}{25200}\left(15625t^7+306250t^6+144375t^5-30625t^4-21000t^3+452t\right) \\ &\beta_{\frac{3}{5}}(t)=\frac{h^2}{5040}(31250t^7+70000t^6+44625t^5-1750t^4-8400t^3+271t) \\ &\beta_{\frac{4}{5}}(t)=\frac{-h^2}{50400}(156250t^7+393750t^6+328125t^5+65625t^4-45500t^3-25200t^2-3104t) \\ &\beta_1(t)=\frac{h^2}{50400}(31250t^7+87500t^6+91875t^5+43750t^4+8400t^3+107t) \end{aligned} \tag{7}$$

Evaluating (3) at $x_n, x_{n+\nu_1}, x_{n+\nu_2}$, and $x_{n+1}$ so that the scaling factor $, t=-\frac{4}{5}, -\frac{3}{5}, -\frac{2}{5}$ and $\frac{1}{5}$ respectively, the following discrete schemes are obtained:

$$3y_{n+\frac{4}{5}}-4y_{n+\frac{3}{5}}+y_n=\frac{h^2}{3000}\left[-f_{n+1}+32f_{n+\frac{4}{5}}+322f_{n+\frac{3}{5}}+232f_{n+\frac{2}{5}}+127f_{n+\frac{1}{5}}+8f_n\right] \quad (8)$$

$$2y_{n+\frac{4}{5}}-3y_{n+\frac{3}{5}}+y_{n+\frac{1}{5}}=\frac{-h^2}{6000}\left[2f_{n+1}-47f_{n+\frac{4}{5}}-412f_{n+\frac{3}{5}}-242f_{n+\frac{2}{5}}-22f_{n+\frac{1}{5}}+f_n\right] \quad (9)$$

$$y_{n+\frac{4}{5}}-2y_{n+\frac{3}{5}}+y_{n+\frac{2}{5}}=\frac{-h^2}{6000}\left[f_{n+1}-24f_{n+\frac{4}{5}}-194f_{n+\frac{3}{5}}-24f_{n+\frac{2}{5}}+f_{n+\frac{1}{5}}\right] \quad (10)$$

$$y_{n+1}-2y_{n+\frac{4}{5}}+y_{n+\frac{3}{5}}=\frac{h^2}{6000}\left[18f_{n+1}+209f_{n+\frac{4}{5}}+4f_{n+\frac{3}{5}}+14f_{n+\frac{2}{5}}-6f_{n+\frac{1}{5}}+f_n\right] \quad (11)$$

Differentiating (3) once gives the derivative formula of the form:

$$h\,y'(x)=\sum_{j'=3,4}\alpha'_{v_{j'}}(t)y_{n+v_{j'}}+h^2\left[\sum_{j=0}^{1}\beta'_j(t)f_{n+j}+\sum_{j'=1}^{4}\beta'_{v_{j'}}(t)f_{n+v_{j'}}\right] \quad (12)$$

Where coefficients $\alpha'_{v_3}(t),\alpha'_{v_4}(t),\beta'_j$ and $\beta'_{v_{j'}}$ are given as follows:

$$\begin{aligned}
&\alpha'_{\frac{3}{5}}(t)=-5\\
&\alpha'_{\frac{4}{5}}(t)=5\\
&\beta'_0(t)=\frac{-h}{10080}(43750t^6+52500t^5+13125t^4-3500t^3-1260t^2+8)\\
&\beta'_{\frac{1}{5}}(t)=\frac{h}{50400}(1093750t^6+1575000t^5+459375t^4-105000t^3-42000t^2+277)\\
&\beta'_{\frac{2}{5}}(t)=-\frac{h}{25200}(1093750t^6+1837500t^5+721875t^4-122500t^3-63000t^2+452)\\
&\beta'_{\frac{3}{5}}(t)=\frac{h}{5040}(218750t^6+420000t^5+223125t^4-7000t^3-25200t^2+271)\\
&\beta'_{\frac{4}{5}}(t)=-\frac{h}{50400}(1093750t^6+2362500t^5+1640625t^4+26500t^3-136500t^2-50400t-3104)\\
&\beta'_1(t)=\frac{h}{50400}(218750t^6+525000t^5+459375t^4+175000t^3+25200t^2-107)
\end{aligned} \quad (13)$$

Evaluating (12) at $x_n, x_{n+\frac{1}{3}}, x_{n+\frac{2}{5}}, x_{n+\frac{3}{5}}, x_{n+\frac{4}{5}}$ and $x_{n+1}$ implies by (6) that $t=-\frac{4}{5},-\frac{3}{5},-\frac{2}{5},0,\frac{1}{5}$. Hence, the following discrete derivative schemes are obtained:

$$h\,y'_n-5y_{n+\frac{4}{5}}+5y_{n+\frac{3}{5}}=\frac{-h^2}{50400}\left[107f_{n+1}+32f_{n+\frac{4}{5}}+11626f_{n+\frac{3}{5}}+6280f_{n+\frac{2}{5}}+14059f_{n+\frac{1}{5}}+3176f_n\right] \quad (14)$$

$$hy'_{n+\frac{1}{5}}-5y_{n+\frac{4}{5}}+5y_{n+\frac{3}{5}}=\frac{h^2}{50400}\left[82f_{n+1}-1243f_{n+\frac{4}{5}}-8252f_{n+\frac{3}{5}}-11866f_{n+\frac{2}{5}}-4070f_{n+\frac{1}{5}}+149f_n\right] \quad (15)$$

$$hy'_{n+\frac{2}{5}}-5y_{n+\frac{4}{5}}+5y_{n+\frac{3}{5}}=\frac{h^2}{50400}\left[-5f_{n+1}+704f_{n+\frac{4}{5}}+10058f_{n+\frac{3}{5}}+4712f_{n+\frac{2}{5}}-389f_{n+\frac{1}{5}}+40f_n\right] \quad (16)$$

$$hy'_{n+\frac{3}{5}}-5y_{n+\frac{4}{5}}+5y_{n+\frac{3}{5}}=\frac{h^2}{50400}\left[82f_{n+1}-1355f_{n+\frac{4}{5}}-4444f_{n+\frac{3}{5}}+902f_{n+\frac{2}{5}}-262f_{n+\frac{1}{5}}+37f_n\right] \quad (17)$$

$$hy'_{n+\frac{4}{5}}-5y_{n+\frac{4}{5}}+5y_{n+\frac{3}{5}}=\frac{h^2}{50400}\left[107f_{n+1}-3104f_{n+\frac{4}{5}}-2710f_{n+\frac{3}{5}}+904f_{n+\frac{2}{5}}-277f_{n+\frac{1}{5}}+40f_n\right] \quad (18)$$

$$hy'_{n+1}-5y_{n+\frac{4}{5}}+5y_{n+\frac{3}{5}}=\frac{h^2}{50400}\left[3218f_{n+1}-13093f_{n+\frac{4}{5}}-2876f_{n+\frac{3}{5}}+2470f_{n+\frac{2}{5}}-934f_{n+\frac{1}{5}}+149f_n\right] \quad (19)$$

***Implementation of the Method:***

The continuous one step implicit hybrid block formula is implemented as a simultaneous integrator which does not need any special method to supply starting values nor predictors, over non over-lapping subintervals; $[x_0, x_1], \ldots, [x_{N-1}, x_N]$ of the partition $\Delta_N : a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$ On implementation, the block formula generates simultaneous solutions at the offgrid and grid points respectively.

A combination of (8)-(11), and (14)-(19) gives the required block structure. Normalizing (1) yields the following constant coefficient matrices, (i.e. replacing *A*, *B*, *C*, *D*) $\tilde{A}$ as a 10 x 10 identity matrix,

$$\tilde{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1231}{12600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{71}{3150} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{123}{3500} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{376}{7875} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{61}{1008} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{19}{288} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{14}{225} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{51}{800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{14}{225} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{19}{288} \end{bmatrix}$$

$$\bar{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{863}{50400}h^2 & \frac{-761}{63000}h^2 & \frac{941}{126000}h^2 & \frac{-341}{126000}h^2 & \frac{107}{252000}h^2 \\ 0 & 0 & 0 & 0 & 0 & \frac{544}{7875}h^2 & \frac{-37}{1575}h^2 & \frac{136}{1875}h^2 & \frac{100}{15750}h^2 & \frac{8}{7875}h^2 \\ 0 & 0 & 0 & 0 & 0 & \frac{350}{28000}h^2 & \frac{-9}{3500}h^2 & \frac{87}{2800}h^2 & \frac{-9}{875}h^2 & \frac{9}{5600}h^2 \\ 0 & 0 & 0 & 0 & 0 & \frac{1424}{1875}h^2 & \frac{176}{7875}h^2 & \frac{608}{7875}h^2 & \frac{-16}{1575}h^2 & \frac{16}{7875}h^2 \\ 0 & 0 & 0 & 0 & 0 & \frac{475}{2016}h^2 & \frac{25}{504}h^2 & \frac{125}{1008}h^2 & \frac{25}{1008}h^2 & \frac{11}{2016}h^2 \\ 0 & 0 & 0 & 0 & 0 & \frac{1427}{7200}h & \frac{-133}{1200}h & \frac{241}{3600}h & \frac{-173}{7200}h & \frac{3}{800}h \\ 0 & 0 & 0 & 0 & 0 & \frac{43}{150}h & \frac{7}{225}h & \frac{7}{225}h & \frac{-1}{75}h & \frac{1}{450}h \\ 0 & 0 & 0 & 0 & 0 & \frac{219}{800}h & \frac{57}{400}h & \frac{57}{400}h & \frac{-21}{800}h & \frac{3}{800}h \\ 0 & 0 & 0 & 0 & 0 & \frac{64}{225}h & \frac{8}{75}h & \frac{64}{225}h & \frac{14}{225}h & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{25}{96}h & \frac{25}{144}h & \frac{25}{144}h & \frac{25}{96}h & \frac{19}{288}h \end{bmatrix}$$

***Analysis of the Method:***

In this section, the order of accuracy, consistency, zero stability, and convergence of the block formula is determined. Also, the interval of absolute stability is given for the continuous implicit one step hybrid block method (1)

***A. Order:***

In the spirit of (Awoyemi, *et al*, 2011) with some modifications we define a linear difference operator associated with the continuous implicit one-step hybrid block method.

***Definition 1:***

*The linear difference operator L associated with (3), evaluated at the grid point (symbol), is defined as*

$$L\left[y(x);h\right] = h^{\tau}\tilde{A}Y_m - h^{\tau}\tilde{B}y_m - h^{\mu-\tau}\left[\tilde{C}F\left(y_m\right) + \tilde{D}F\left(Y_m\right)\right] \tag{20}$$

*Where* $y(x)$ *is an arbitrary test function continuously differentiable on* $[a,b]$. *Expanding* $Y_m$ *and* $F(Y_m)$ *component wise respectively; nothing that* $f_{n+j} = y''_{n+j}$ , $j = 0, v_{i-3}, v_{i-2}, v_{i-1}, v_i, 1$; *in Taylor series and collecting terms in powers of h gives*

$$L\left[y(x);h\right] = \bar{C}_0 y(x) + \bar{C}_1 h\, y'(x) + \bar{C}_2 h^2 y^{(2)}(x) + \cdots + C_p h^p y^{(p)}(x) + C_{p+1} h^{p+1} y^{(p+1)}(x) + \ldots \tag{21}$$

*Where the* $\bar{C}_r, r = 0, 1, \ldots$, *are vectors.*

***Definition 2:***

*The associated linear difference operator (10) and the continuous implicit one-step hybrid block method (1) are said to have order equals to p if* $\bar{C}_0=\bar{C}_1=\cdots=\bar{C}_p=\bar{C}_{p+1}=0$ *and* $\bar{C}_{p+2}\neq 0$ .

***Definition 3:***

*The term* $\bar{C}_{p+2}$ *is called the error constant. It implies that the continuous implicit one step hybrid block method (1) has local truncation error (lte) given by*

$$L\left[y(x);h\right]=\bar{C}_{p+2}h^{p+2}y^{p+2}\left(x_n\right)+O(h^{P+3}) \tag{22}$$

Our computation shows that the continuous implicit one step hybrid block method has order of accuracy $p=\left(6,6,6,6,6,6,6,6,6,6\right)^T$ with local error constant

$$\bar{C}_{p+2}=\left(\tfrac{-199}{49500000000},\tfrac{-19}{369140625},\tfrac{-141}{1750000000},\tfrac{-8}{73828125},\tfrac{-11}{75600000},\tfrac{-863}{4725000000},\tfrac{-37}{295312500},\tfrac{-29}{175000000},\tfrac{-8}{73828125},\tfrac{-11}{37800000}\right)^T$$

***B Consistency:***

Clearly, the consistency of the continuous implicit one step hybrid block method can be established from the fact that it has order $p>1$, (Lambert, 1973).

***C Zero Stability:***

***Definition 4:***

*The method (1), is said to be zero stable as* $h\to 0$ *if its first characteristic polynomial* $\rho(z)$ *satisfies*

$$\rho(z)=\det\left[z\tilde{A}-\tilde{B}\right]=z^{q-\mu}(z-1)^{\mu}=0 \tag{23}$$

*where* $q$ *represents the order of the matrices* $\tilde{A}$ *and* $\tilde{B}$*; and the roots* $Z_s,\ s=1,\ldots,10$ *of (12) satisfy the conditions* $|Z_s|\leq 1$*. Furthermore, those roots with* $|Z_s|=1$ *have multiplicity not exceeding the order of the differential equation.*

The continuous implicit one step hybrid block method satisfies the conditions in Definition 8 since from (1), for $q=10$ and $\mu=2$;

$$\det\left[z\tilde{A}-\tilde{B}\right]=z^8\left(z-1\right)^2=0$$

Hence, the method is zero stable.

***D Convergence:***

Following (Henrici, 1962), the convergence of the continuous one step hybrid block method is established since the method is consistent and zero stable.

***E Absolute Stability:***

In what follows, absolute stability will be discussed using the boundary locus method. Consider the stability polynomial

$$\Pi\left(z,\bar{h}\right)=\rho(z)-\bar{h}\sigma(z)=0 \tag{24}$$

Where $\bar{h}=h^2\lambda^2$ and $\lambda=\frac{df}{dy}$ are assumed constant.

The polynomial (24) is obtained by applying the continuous implicit one step hybrid method (3) to the scalar test problem:

$$y''=-\lambda^2 y \tag{25}$$

The stability polynomial of continuous implicit one step hybrid block method (1) is obtained similarly by applying the scalar test problem (25) to the block formula. We shall be guided by the following definitions.

***Definition 5:***

*The method (1) is said to be absolutely stable if for a given* $\bar{h}$ *all the roots* $z_s$ *of (24) satisfy* $|z_s| < 1,\ s = 1, 2, \ldots, n.$

***Definition 6:***

*The region* $\Re$ *of the complex* $\bar{h}$*-plane such that the roots of* $\Pi(z, \bar{h}) = 0$ *lie within the unit circle whenever* $\bar{h}$ *lies in the interior of the region is called the region of absolute stability.*

Now, after obtaining the first and second characteristics polynomials for the block method, we have

$$\tilde{A}Y_m = \tilde{B}y_m + (\lambda h)^2 \left[ \tilde{C}F(y_m) + \tilde{D}F(Y_m) \right] \tag{26}$$

reduced to the locus of the boundary of the region of absolute stability as follows:

$$\bar{H}(z,h) = \left[ \tilde{C}(z) + \tilde{D}(z) \right]^{-1} \left[ \tilde{A}(z) - \tilde{B}(z) \right] \tag{27}$$

Where $\bar{H}(z,h) = (\lambda h)^2$. Setting $z = e^{i\theta},\ 0 \le \theta \le \pi$ in (27) yields the stability matrix

$$\bar{H}(\theta,h) = \left[ \tilde{C}\left(e^{i\theta}\right) + \tilde{D}\left(e^{i\theta}\right) \right]^{-1} \left[ \tilde{A}\left(e^{i\theta}\right) - \tilde{B}\left(e^{i\theta}\right) \right] \tag{1.1}$$

Note that the determinant of $\bar{H}(\theta,h)$ is the stability polynomial of the block method (1)

After a little computation, the interval of absolute stability obtained for the continuous one step implicit block method is $(0, -60,000)$, the region of absolute stability is shown in Figure 1:

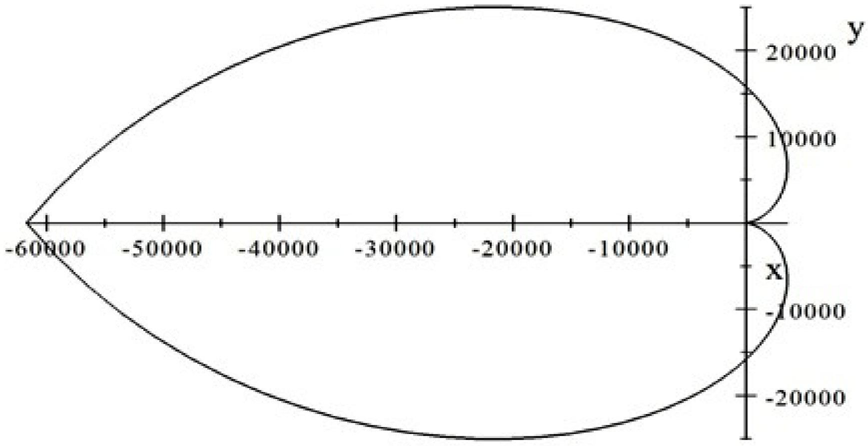


**Fig. 1:** Region of absolute stability of the implicit one step block method

***Numerical Experiment:***

The proposed method is experimented with two problems to test its accuracy on the basis of step number, order and absolute error. In each case, the computed result obtained is compared to results obtained for the same problems computed with existing methods.

***Problem 1:***

$$y'' - x(y')^2 = 0, \quad y(0) = 1, \quad y'(0) = \frac{1}{2}, \ h = \frac{0.1}{32}$$

$$Exact\ Solution: y = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$$

***Problem 2:***

$$y'' + \left(\frac{6}{x}\right)y' + \left(\frac{4}{x^2}\right)y = 0, \ y(1) = y'(1) = 1, \ h = \frac{0.1}{32}$$

$$Exact\ Solution: \frac{5}{3x} - \frac{3}{3x^4}$$

**Table I:** for problem 1

| X | Error in Jator and Li , (2009) | Error in Anake *et al* , (2012) | Error in New result |
|---|---|---|---|
| 0.1 | 0.11839E-09 | 0.49827E-10 | 0.75273E-13 |
| 0.2 | 0.23750E-09 | 0.41043E-09 | 0.58908E-13 |
| 0.3 | 0.42485E-09 | 0.14286E-08 | 0.19071E-12 |
| 0.4 | 0.61629E-09 | 0.35243E-08 | 0.42331E-11 |
| 0.5 | 0.10233E-08 | 0.72435E-08 | 0.79977E-11 |
| 0.6 | 0.14484E-08 | 0.13336E-07 | 0.11119E-10 |
| 0.7 | 0.25450E-08 | 0.22873E-07 | 0.13619E-10 |
| 0.8 | 0.37221E-08 | 0.37447E-07 | 0.11938E-10 |
| 0.9 | 0.73288E-08 | 0.59504E-07 | 0.16098E-12 |
| 1.0 | 0.11338E-08 | 0.92940E-07 | 0.32906E-10 |

**Table II:** for problem 2

| X | Error in Badmus and Yahaya(2009) | Error in New Result |
|---|---|---|
| 1.0031 | 0.383540E-04 | 0.5756500E-11 |
| 1.0063 | 0.750040E-04 | 0.2257883E-10 |
| 1.0094 | 0.105923E-03 | 0.5249712E-10 |
| 1.0125 | 0.135480E-03 | 0.9222134E-10 |
| 1.0156 | 0.155570E-03 | 0.1421980E-09 |
| 1.0188 | 0.186370E-03 | 0.2019609E-09 |
| 1.0219 | 0.196060E-03 | 0.2710634E-09 |
| 1.0250 | 0.221040E-03 | 0.3490721E-09 |
| 1.0281 | 0.205630E-03 | 0.4355714E-09 |
| 1.0313 | 0.277910E-03 | 0.5301604E-09 |

### *Conclusion:*

The continuous implicit one step hybrid block method is developed from combinations of continuous implicit one step schemes and implemented as a simultaneous integrator over non overlapping intervals. The accuracy of the method compared to existing methods is seen in Tables I and II. The convergent order six method is suitable for the solutions of initial value problems of general second order differential equations.

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