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Nonlinear integro-differential equations by differential transform method with Adomian polynomials

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Abstract: A modification of differential transformation method is applied to nonlinear integro-differential equations. In this technique, the nonlinear term is replaced by its Adomian polynomials for the index k, and hence the dependent variable components are replaced in the recurrence relation by their corresponding differential transform components of the same index. Thus the nonlinear integro-differential equation can be easily solved with less computational work for any analytic nonlinearity due to the properties and available algorithms of the Adomian polynomials. New theorems for products and integrals with nonlinear functions are introduced. Several illustrative examples with different types of nonlinearities are considered to indicate the effectiveness of the present technique.

***Key words*:** Differential transform method; nonlinear integro-differential equations; Adomian polynomials.

INTRODUCTION

Integral and integro-differential equations play an important role in characterizing many social, biological, physical and engineering problems; for more details see (Kythe and Puri, 1992; Wazwaz, 2006; Rashed, 2004) and references cited therein. Nonlinear integral and integro-differential equations are usually hard to solve analytically and exact solutions are rather difficult to be obtained. In literature there exist many numerical methods have been studied such as Haar wavelets method (Aziz and Islam, 2013), the rationalized Haar functions method (Maleknejed and Mirzaee, 2006; Reihani and Abadi, 2007), the linearization method (Darania *et al*., 2006), the finite difference method (Zhao and Corless, 2006), the Tau method (Abbasbandy and Taati, 2009; Ebadi *et al*., 2007), the hybrid Legendre polynomials and block-pulse functions (Maleknejad *et al*., 2011), the Adomian decomposition method (Wazwaz, 2010; Araghi and Behzadi, 2009), the Taylor polynomial method (Darania and Ivaz, 2008; Maleknejad and Mohmoudi, 2003; Yalcinbas, 2012) and the differential transform method (Borhanifar and Abazari, 2012).

The differential transform method (DTM) has been proved to be efficient for handling nonlinear problems, but the nonlinear functions used in these studies are restricted to polynomials and products with derivatives (Borhanifar and Abazari, 2012; Arikoglu and Ozkol, 2005; Arikoglu and Ozkol, 2008; Odibat, 2008; Biazar and Eslami, 2011). For other types of nonlinearities, the usual way to calculate their transformed functions as introduced by (Zhou, 1986) is to expand the nonlinear function in an infinite power series then take the differential transform of this series. The problem with this approach is that the massive computational difficulties will arise in determining the differential transform of nonlinear function while working with this infinite series. Another approach for obtaining the differential transform of nonlinear terms is the algorithm in (Chang and Chang, 2008). It is based on using the properties of differential transform and calculus to develop a canonical equation. Then this equation is solved for the required differential transform of nonlinear term. But, as seen in the simple examples in section 3 in (Chang and Chang, 2008), the algorithm requires a sequence of differentiation, algebraic manipulations and computations of differential transform for other functions which is more difficult for the case of composite nonlinearities.

In this work, we introduce a comprehensive and more efficient approach for using the DTM to solve nonlinear integro-differential equations; the idea is based on the methodology in (Elsaid, 2012). The nonlinear function is replaced by its Adomian polynomials and then the dependent variable components are replaced by their corresponding differential transform component of the same index. This technique benefits the properties of the Adomian polynomials and the efficient algorithm to generate them quickly as in the work (Duan, 2010, 2011). Numerical simulations of integro-differential equations with different types of nonlinearity are treated and the proposed technique has provided good results.

2. Differential Transform Method:

The basic definition and the fundamental theorems of the differential transformation and its applicability for various kinds of differential and integral equations are given in (Arikoglu and Ozkol, 2005; Arikoglu and Ozkol, 2008; Odibat, 2008; Biazar and Eslami, 2011; Zhou, 1986). For convenience of the reader, a review of

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differential transformation will be presented here. The transformation of the *k*-th derivative of a function in one variable is as follows.

$$Y(k)=\frac{1}{k!}\left[\frac{d^k}{dx^k}y(x)\right]_{x=x_0}, \tag{1}$$

and the inverse transformation is defined by

$$y(x)=\sum_{k=0}^{\infty}Y(k)(x-x_0)^k. \tag{2}$$

The following theorems can be deduced from equations (1) and (2).

Theorem1. If $y(x)=f(x)\pm h(x)$, then $Y(k)=F(k)\pm H(k)$.

Theorem2. If $y(x)=cf(x)$, then $Y(k)=cF(k)$, where c is a constant.

Theorem3. If $y(x)=f^{(n)}(x)$, then $Y(k)=\frac{(k+n)!}{k!}F(k)$.

Theorem4. If $y(x)=f(x)h(x)$, then $Y(k)=\sum_{k_1=0}^{k}F(k_1)H(k-k_1)$.

Theorem5. If $y(x)=x^m$, then $Y(k)=\delta(k-m)$, where $\delta(k-m)=\begin{cases}1, & k=m\\ 0, & k\neq m\end{cases}$.

3. The Modified Differential Transform Method:

In this section, we will introduce a reliable and efficient algorithm to calculate the differential transform of a nonlinear function $g(y(x))$. The Adomian polynomials of this nonlinear function are determined formally as follows (Wazwaz, 2010; Biazar *et al*., 2009)

$$\widetilde{A}_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[g\left(\sum_{i=0}^{\infty}\lambda_i y_i\right)\right]\right]_{\lambda=0}, n\geq 0.$$

That is, the Adomian polynomials of $g(y(x))$ are

$$\widetilde{A}_0=g(y_0),$$

$$\widetilde{A}_1=y_1g^{(1)}(y_0),$$

$$\widetilde{A}_2=y_2g^{(1)}(y_0)+(1/2!)y_1^2g^{(2)}(y_0),$$

$$\widetilde{A}_3=y_3g^{(1)}(y_0)+y_1y_2g^{(2)}(y_0)+(1/3!)y_1^3g^{(3)}(y_0),$$

$$\widetilde{A}_4=y_4g^{(1)}(y_0)+(y_1y_3+(1/2!)y_2^2)g^{(2)}(y_0)+(1/2!)y_1^2y_2g^{(3)}(y_0)+(1/4!)y_1^4g^{(4)}(y_0),$$

$$\widetilde{A}_5=y_5g^{(1)}(y_0)+(y_2y_3+y_1y_4)g^{(2)}(y_0)+(1/2!)(y_1^2y_3+y_1y_2^2)g^{(3)}(y_0)$$

$$+(1/3!)y_1^3y_2g^{(4)}(y_0)+(1/5!)y_1^5g^{(5)}(y_0),$$ and so on.

Lemma: If $f(x)=g(y(x))$, then $F(k)=A_k$ where A_k are the Adomian polynomials $\widetilde{A}_k$ but with replacing y_k by $Y(k)$, $k=0,1,2,\cdots$.

***Proof*:** The differential transforms of $f(x)$ are computed by utilizing (1) as

$$F(0)=\frac{1}{0!}\{g(y(x))\}_{x=x_0}=g(y(x_0))=g(Y(0))=A_0,$$

$$F(1)=\frac{1}{1!}\left\{\frac{d}{dx}g(y(x))\right\}_{x=x_0}=y^{(1)}(x_0)g^{(1)}(y(x_0))=Y(1)g^{(1)}(Y(0))=A_1,$$

$$F(2)=\frac{1}{2!}\left\{\frac{d^2}{dx^2}g(y(x))\right\}_{x=x_0}=\frac{1}{2!}\left\{y^{(2)}(x_0)g^{(1)}(y(x_0))+(y^{(1)}(x_0))^2g^{(2)}(y(x_0))\right\}$$

$$=Y(2)g^{(1)}(Y(0))+\frac{1}{2!}(Y(1))^2g^{(2)}(Y(0))=A_2,$$

$$F(3)=\frac{1}{3!}\left\{\frac{d^3}{dx^3}g(y(x))\right\}_{x=x_0}$$

$$=\frac{1}{3!}\left\{y^{(3)}(x_0)g^{(1)}(y(x_0))+3y^{(1)}(x_0)y^{(2)}(x_0)g^{(2)}(y(x_0))+(y^{(1)}(x_0))^3g^{(3)}(y(x_0))\right\}$$

$$=Y(3)g^{(1)}(Y(0))+Y(1)Y(2)g^{(2)}(Y(0))+\frac{1}{3!}(Y(1))^3g^{(3)}(Y(0))=A_3.$$

In general we have, $F(k)=A_k$. □

Consequently, the inverse transform of the nonlinear function can be written as

$$f(x)=g(y(x))=\sum_{k=0}^{\infty}A_k(x-x_0)^k, \tag{3}$$

where, A_k are the differential transform of $f(x)=g(y(x))$.

The advantage of using this algorithm for computing differential transformation of nonlinear functions comparing with the algorithm suggested in (Chang and Chang, 2008), is this algorithm dealing directly with nonlinear function of the problem in hand in its form without any differentiation or algebraic manipulations or even there is no need to compute the differential transform of other functions to obtain the required one. This will be clear throughout the following theorems.

Theorem 6. If $f(x)=h(x)g(y(x))$, then $F(k)=\sum_{k_1=0}^{k}H(k_1)A_{k-k_1}$.

Proof: By utilizing definition (1), we can get

$$F(0)=\frac{1}{0!}\{h(x)g(y(x))\}_{x=x_0}=h(x_0)g(y(x_0))=H(0)g(Y(0))=H(0)A_0,$$

$$F(1)=\frac{1}{1!}\left\{\frac{d}{dx}[h(x)g(y(x))]\right\}_{x=x_0}$$

$$=h^{(1)}(x_0)g(y(x_0))+h(x_0)y^{(1)}(x_0)g^{(1)}(y(x_0))=H(1)A_0+H(0)A_1$$

$$F(2)=\frac{1}{2!}\left\{\frac{d^2}{dx^2}[h(x)g(y(x))]\right\}_{x=x_0}=\frac{1}{2!}\left\{h^{(2)}(x_0)g(y(x_0))+2h^{(1)}(x_0)y^{(1)}(x_0)g^{(1)}(y(x_0))\right.$$

$$\left.+h(x_0)[y^{(2)}(x_0)g^{(1)}(y(x_0))+(y^{(1)}(x_0))^2g^{(2)}(y(x_0))]\right\}$$

$$=H(2)A_0+H(1)A_1+H(0)A_2,$$

$$F(3)=\frac{1}{3!}\left\{\frac{d^3}{dx^3}[h(x)g(y(x))]\right\}_{x=x_0}=\frac{1}{3!}\left\{h^{(3)}(x_0)g(y(x_0))+3h^{(2)}(x_0)y^{(1)}(x_0)g^{(1)}(y(x_0))\right.$$

$$+3h^{(1)}(x_0)[y^{(2)}(x_0)g^{(1)}(y(x_0))+(y^{(1)}(x_0))^2g^{(2)}(y(x_0))]+h(x_0)[y^{(3)}(x_0)g^{(1)}(y(x_0))$$

$$\left.+3y^{(1)}(x_0)y^{(2)}(x_0)g^{(2)}(y(x_0))+(y^{(1)}(x_0))^3g^{(3)}(y(x_0))]\right\}$$

$$=H(3)A_0+H(2)A_1+H(1)A_2+H(0)A_3.$$

In general we have, $F(k)=\sum_{k_1=0}^{k}H(k_1)A_{k-k_1}$. □

Theorem 7. If $f(x)=\int_{x_0}^{x}g(y(t))dt$, then $F(k)=\frac{A_{k-1}}{k}, k\geq 1$.

***Proof*:** By using (3), the transform of the integral can be found as

$$f(x)=\int_{x_0}^{x}\sum_{k=0}^{\infty}A_k(t-x_0)^k\,dt=\sum_{k=0}^{\infty}A_k\int_{x_0}^{x}(t-x_0)^k\,dt=\sum_{k=1}^{\infty}\frac{A_{k-1}}{k}(t-x_0)^k.$$

Again utilizing (3), we get $F(k)=\frac{A_{k-1}}{k}$, where $k\geq 1$ and $F(0)=f(x_0)=0$. □

Theorem 8. If $f(x)=h(x)\int_{x_0}^{x}g(y(t))dt$, then $F(k)=\sum_{k_1=1}^{k}\frac{1}{k_1}H(k-k_1)A_{k_1-1}$, $k\geq 1$.

***Proof*:** Utilizing the definition of the transform, we can get

$$F(0)=\frac{1}{0!}\left\{h(x)\int_{x_0}^{x}g(y(t))dt\right\}_{x=x_0}=0,$$

$$F(1)=\frac{1}{1!}\left\{\frac{d}{dx}\left[h(x)\int_{x_0}^{x}g(y(t))dt\right]\right\}_{x=x_0}$$

$$=\left\{h^{(1)}(x)\int_{x_0}^{x}g(y(t))dt+h(x)g(y(x))\right\}_{x=x_0}=h(x_0)g(y(x_0))$$

$$=H(0)A_0,$$

$$F(2)=\frac{1}{2!}\left\{\frac{d^2}{dx^2}\left[h(x)\int_{x_0}^{x}g(y(t))dt\right]\right\}_{x=x_0}=\frac{1}{2!}\left\{2h^{(1)}(x_0)g(y(x_0))+h(x_0)y^{(1)}(x_0)g^{(1)}(y(x_0))\right\}$$

$$=H(1)A_0+H(0)A_1/2,$$

$$F(3)=\frac{1}{3!}\left\{\frac{d^3}{dx^3}\left[h(x)\int_{x_0}^{x}g(y(t))dt\right]\right\}_{x=x_0}$$

$$=\frac{1}{3!}\left\{3h^{(2)}(x_0)g(y(x_0))+3h^{(1)}(x_0)y^{(1)}(x_0)g^{(1)}(y(x_0))\right.$$

$$\left.+h(x_0)[y^{(2)}(x_0)g^{(1)}(y(x_0))+(y^{(1)}(x_0))^2g^{(2)}(y(x_0))\right\}=H(2)A_0+\frac{1}{2}H(1)A_1+\frac{1}{3}H(0)A_2.$$

In general we have, $F(k)=\sum_{k_1=1}^{k}\frac{1}{k_1}H(k-k_1)A_{k_1-1}$, where $k\geq 1$. □

Theorem 9. If $f(x)=\int_{x_0}^{x}g_1(t)g_2(y(t))dt$, then $F(k)=\frac{1}{k}\sum_{k_1=0}^{k-1}G_1(k_1)A_{k-k_1-1}$, $k\geq 1$.

***Proof*:** Utilizing the definition of the transform, we can get

$$F(0)=\frac{1}{0!}\left\{\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right\}_{x=x_0}=0,$$

$$F(1)=\frac{1}{1!}\left\{\frac{d}{dx}\left[\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right]\right\}_{x=x_0}=g_1(x_0)g_2(y(x_0))=G_1(0)A_0,$$

$$F(2)=\frac{1}{2!}\left\{\frac{d^2}{dx^2}\left[\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right]\right\}_{x=x_0}$$

$$=\frac{1}{2!}\left\{g_1^{(1)}(x_0)g_2(y(x_0))+g_1(x_0)y^{(1)}(x_0)g_2^{(1)}(y(x_0))\right\}=[G_1(1)A_0+G_1(0)A_1]/2,$$

$$F(3)=\frac{1}{3!}\left\{\frac{d^3}{dx^3}\left[\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right]\right\}_{x=x_0}=\frac{1}{3!}\left\{g_1^{(2)}(x_0)g_2(y(x_0))+2g_1^{(1)}(x_0)y^{(1)}(x_0)g_2^{(1)}(y(x_0))\right.$$

$$\left.+g_1(x_0)[y^{(2)}(x_0)g_2^{(1)}(y(x_0))+(y^{(1)}(x_0))^2g_2^{(2)}(y(x_0))]\right\}$$

$$=[G_1(2)A_0+G_1(1)A_1+G_1(0)A_2]/3.$$

In general we have, $F(k)=\frac{1}{k}\sum_{k_1=0}^{k-1}G_1(k_1)A_{k-k_1-1}$, where $k\ge 1$. □

Theorem 10. If $f(x)=h(x)\int_{x_0}^{x} g_1(t)g_2(y(t))dt$, then

$$F(k)=\sum_{k_2=1}^{k}\sum_{k_1=1}^{k_2}\frac{1}{k_2}G_1(k_1-1)A_{k_2-k_1}H(k-k_2).$$

***Proof*:** Utilizing the definition of the transform, we can get

$$F(0)=\frac{1}{0!}\left\{h(x)\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right\}_{x=x_0}=0,$$

$$F(1)=\frac{1}{1!}\left\{\frac{d}{dx}\left[h(x)\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right]\right\}_{x=x_0}=h(x_0)g_1(x_0)g_2(y(x_0))=H(0)G_1(0)A_0,$$

$$F(2)=\frac{1}{2!}\left\{\frac{d^2}{dx^2}\left[h(x)\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right]\right\}_{x=x_0}$$

$$=\frac{1}{2!}\left\{2h^{(1)}(x_0)g_1(x_0)g_2(y(x_0))+h(x_0)[g_1^{(1)}(x_0)g_2(y(x_0))+g_1(x_0)y^{(1)}(x_0)g_2^{(1)}(y(x_0))]\right\}$$

$$=H(1)G_1(0)A_0+[H(0)G_1(1)A_0+H(0)G_1(0)A_1]/2,$$

$$F(3)=\frac{1}{3!}\left\{\frac{d^3}{dx^3}\left[h(x)\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right]\right\}_{x=x_0}$$

$$=\frac{1}{3!}\Big\{3h^{(2)}(x_0)g_1(x_0)g_2(y(x_0))+3h^{(1)}(x_0)[g_1^{(1)}(x_0)g_2(y(x_0))+g_1(x_0)y^{(1)}(x_0)g_2^{(1)}(y(x_0))]$$

$$+h(x_0)[g_1^{(2)}(x_0)g_2(y(x_0))+2g_1^{(1)}(x_0)y^{(1)}(x_0)g_2^{(1)}(y(x_0))]$$

$$+g_1(x_0)[y^{(2)}(x_0)g_2^{(1)}(y(x_0))+(y^{(1)}(x_0))^2 g_2^{(2)}(y(x_0))]\Big\}$$

$$=H(2)G_1(0)A_0+[H(1)G_1(1)A_0+H(1)G_1(0)A_1]/2+[H(0)G_1(2)A_0+H(0)G_1(1)A_1+H(0)G_1(0)A_2]/3$$

In general we have, $F(k)=\sum_{k_2=1}^{k}\sum_{k_1=1}^{k_2}\frac{1}{k_2}G_1(k_1-1)A_{k_2-k_1}H(k-k_2)$. □

The following relation is quite useful in the solution of Fredholm integrals; it can be obtained from theorem 9 and equation (2)

$$\int_a^b g_1(t)g_2(y(t))dt=\sum_{k_1=1}^{\infty}\left\{\frac{1}{k}\left[(b-x_0)^k-(a-x_0)^k\right]\sum_{k_1=0}^{k-1}G_1(k_1)A_{k-k_1-1}\right\}.$$

(4)

4. Applications and Numerical Results

In this section, we implement the proposed method on some different examples with different types of nonlinearity.

Example 1. Consider the nonlinear Volterra integro-differential equation

$$y''(x)+y'(x)y(x)+y(x)=\cos 2x+x^3-x^2\int_0^x\frac{1+\sin 2t}{y^2(t)}dt\ ,0\le x\le 1\ , \tag{5}$$

with the initial conditions

$$y(0)=1 \text{ and } y'(0)=1\ . \tag{6}$$

The differential transformation of equation (5) and the initial conditions (6) are

$$Y(k+2)=\frac{k!}{(k+2)!}\left[\frac{2^k}{k!}\cos(\pi k/2)+\delta(k-3)-\sum_{m=0}^{k}(m+1)Y(m+1)Y(k-m)-Y(k)\right.$$

$$\left.-\frac{A_{k-3}}{k-2}-\frac{1}{k-2}\sum_{m=1}^{k-2}\frac{2^{m-1}}{(m-1)!}\sin\left(\frac{\pi(m-1)}{2}\right)A_{k-m-2}\right],$$

where $[2^k\cos(\pi k/2)]/k!$ and $[2^k\sin(\pi k/2)]/k!$ are the differential transforms of $\cos(2x)$ and $\sin(2x)$, respectively and A_k are the differential transform of the nonlinear function $g(y)=y^{-2}$, and $Y(0)=Y(1)=1$. Using the Lemma, A_k are: $A_0=g(Y(0))=1$, $A_1=-2Y(1)$, $A_2=-2Y(2)+3Y^2(1)$, $A_3=-2Y(3)+6Y(1)Y(2)-4Y^3(1)$,

$A_4=-2Y(4)-2Y(1)Y(3)-Y^2(2)+3Y^2(1)Y(2)+5Y^4(1)$, … .

Utilizing the recurrence relation, the transformed initial conditions and A_k , $Y(k)$ are evaluated. Hence using the inverse transformation formula, the following series solution up to $O(x^{10})$ can be obtained

$$y(x)=1+x-\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}-\frac{x^6}{6!}-\frac{x^7}{7!}+\frac{x^8}{8!}+\frac{x^9}{9!}+O(x^{10})\ .$$

For sufficiently large number of terms, the closed form of the solution is $y(x) = \sin x + \cos x$, which is the exact solution. Table 1 shows the absolute relative error obtained for three various numbers of terms and at some test points.

Table 1: *Numerical comparison of results in example 1.*

x	*Abs. rel. err.*, (5 Terms)	*Abs. rel. err.*, (10 Terms)	*Abs. rel. err.*, (15 Terms)
0.2	7.75099E-08	0	0
0.4	4.5762E-06	7.74161E-13	0
0.6	5.0297E-05	6.19746E-11	0
0.8	0.000283721	1.4145E-09	1.25621E-15

Example 2. Consider the nonlinear Volterra integro-differential equation

$$y''(x) - 6y(x) = -4 + 8\int_0^x t y(t) \ln y(t) dt \quad 0 \le x \le 1, \tag{7}$$

with the initial conditions

$$y(0) = 1 \text{ and } y'(0) = 0. \tag{8}$$

Application of the differential transform to equation (7) gives

$$(k+1)(k+2)Y(k+2) - 6Y(k) = -4\delta(k) + (8/k)A_{k-2}, \tag{9}$$

where A_k are obtained from Adomian polynomials for the nonlinear function $g(y) = y \ln y$.

Substitute $k = 0$ and 1 into equation (5), one can get the following relations

$$2Y(2) - 6Y(0) = -4, \tag{10}$$

$$6Y(3) - 6Y(1) = 0. \tag{11}$$

Also, for $k \ge 2$ equation (9) can be written as

$$Y(k+2) = \frac{1}{(k+1)(k+2)}\left[6Y(k) + \frac{8}{k}A_{k-2}\right]. \tag{12}$$

The initial conditions in equations (8) is transformed by using (1) to

$$Y(0) = 1 \text{ and } Y(1) = 0. \tag{13}$$

The Adomian polynomials for the nonlinear function of this example are

$$A_0 = g(Y(0)) = 1,\ A_1 = Y(1),\ A_2 = Y(2) - (1/2)Y^2(1),\ A_3 = Y(3) - Y(1)Y(2) + (1/3)Y^3(1)$$

$$A_4 = Y(4) - Y(1)Y(3) - (1/2)Y^2(2) + Y^2(1)Y(2) - (1/4)Y^4(1),\ A_5 = Y(5) - Y(2)Y(3) - Y(1)Y(4)$$

$$+ Y^2(1)Y(3) + Y(1)Y^2(2) - Y^3(1)Y(2) + (1/5)Y^5(1),$$

$$A_6 = Y(6) - (1/2)Y^2(3) - Y(2)Y(4) - Y(1)Y(5)$$

$$+ (1/2)Y^2(1)Y^2(2) + (1/3)Y^3(1)Y(3) - (3/2)Y^2(1)Y^2(2) - Y^3(1)Y(3) + Y^4(1)Y(2) - (1/6)Y^6(1),$$

….

Utilizing the above relations for A_k and relations (10)-(13), one can easily solve for $Y(k)$. Using the inverse transformation rule (2), the following series solution for this problem up to $O(x^{11})$ is

$$y(x) = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \frac{1}{120}x^{10} + O(x^{11}).$$

For sufficiently large number of terms, the closed form of the solution is $y(x) = e^{x^2}$, which is the exact solution. Table 2 shows the absolute relative error obtained for three various numbers of terms and at some test points.

Table 2: *Numerical comparison of results in example 2.*

x	*Abs. rel. err.*, (5 Terms)	*Abs. rel. err.*,(10 Terms)	*Abs. rel. err.*, (15 Terms)
0.2	8.25371E-10	0	0
0.4	7.64934E-07	2.83821E-15	0
0.6	3.73777E-05	7.2663E-12	0
0.8	0.000527137	1.77822E-09	0

Example 3. Consider the nonlinear Volterra integro-differential equation

$$6(x^2+1)y'(x)=(x^3+3x^2+6x+6)e^{-x}+\int_0^x t^3 e^{-\tan y(t)}dt, \quad 0\le x\le 1, \tag{14}$$

with the initial condition

$$y(0)=0. \tag{15}$$

The differential transformation of equation (14) and the initial condition (15) are

$$Y(k+1)=-\frac{k-1}{k+1}Y(k-1)+\frac{1}{6(k+1)}\left[\frac{(-1)^k(6-11k+6k^2-k^3)}{k!}+\frac{A_{k-4}}{k}\right], \tag{16}$$

where $\lambda^k/k!$ are the differential transforms of $e^{\lambda x}$, A_k are the differential transforms the nonlinear function $g(y)=e^{-\tan y}$ and $Y(0)=0$. If we put $x=0$ into equation (14), we can get $y'(0)=1$ and hence $Y(1)=1$.

The following system for $k=1,2,3,\cdots,8$ is obtained from (16)

$$Y(2)=0\quad,\quad Y(3)=-\frac{1}{3}Y(1)\quad,\quad Y(4)=-\frac{1}{2}Y(2),\quad Y(5)=-\frac{3}{5}Y(3)-\frac{1}{5!}+\frac{A_0}{120},$$

$$Y(6)=-\frac{2}{3}Y(4)+\frac{4}{6!}+\frac{A_1}{180},$$

$$Y(7)=-\frac{5}{7}Y(5)-\frac{10}{7!}+\frac{A_2}{252},\ Y(8)=-\frac{3}{4}Y(6)+\frac{20}{8!}+\frac{A_3}{336},\ Y(9)=-\frac{7}{9}Y(7)-\frac{35}{9!}+\frac{A_4}{432},$$

where differential transform components A_k are: $A_0=e^{-\tan(Y(0))}=1$, $A_1=Y(1)$,

$A_2=Y^2(1)-Y(2)$, $A_3=-(1/2)Y^3(1)+Y(1)\ Y(2)\ -Y(3)$,

$A_4=Y(1)Y(3)+(1/2)Y^2(2)-(3/2)Y^2(1)Y(2)+(3/8)Y^4(1)-Y(4)$.

By solving the above systems for $Y(k)$, the series solution of problem (14) and (15) up to $O(x^{10})$ is given by

$$y(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}+O(x^{10}).$$

For sufficiently large of terms, the closed form of the solution is $y(x)=\tan^{-1}x$, which is the exact solution. Table 3 shows the absolute relative error obtained for three various numbers of terms and at some test points.

Table 3: *Numerical comparison of results in example 3.*

X	*Abs. rel. err.*, (5 Terms)	*Abs. rel. err.*,(10 Terms)	*Abs. rel. err.*, (15 Terms)
0.2	9.12337E-09	0	0
0.4	8.82908E-06	4.80303E-10	0
0.6	0.000468447	1.45549E-06	5.92E-10
0.8	0.007532013	0.000411299	2.95854E-07

Example 4. Let us consider the nonlinear Volterra integro-differential equation

$$y''(x)-2y(x)y'(x)=-x+\int_0^x\frac{y'(t)}{1+y^2(t)}dt,\ 0\le x\le 1, \tag{17}$$

with the initial conditions

$$y(0)=0,\ \text{and}\ y'(0)=1. \tag{18}$$

The differential transformation of this equation and its initial conditions are

$$Y(k+2)=\frac{k!}{(k+2)!}\left[2\sum_{m=0}^{k}(m+1)Y(m+1)Y(k-m)-\delta(k-1)+\frac{1}{k}\sum_{m=0}^{k-1}(m+1)Y(m+1)A_{k-m-1}\right],$$

$Y(0)=0$ and $Y(1)=1$.

A_k can be obtained by using Lemma as: $A_0=(1+Y^2(0))^{-1}=1$, $A_1=0$, $A_2=-Y^2(1)$, $A_3=-2Y(1)Y(2)$, $A_4=-2(Y(1)Y(3)+Y^2(2))+Y^4(1)$, … .

By solving for $Y(k)$, the series solution of problem (17) and (18) up to $O(x^{10})$ is given by

$$y(x)=x+x^3/3+2x^5/15+17x^7/315+62x^9/2835+O(x^{10}).$$

For sufficiently large number of terms, the closed form of the solution is $y(x)=\tan x$, which is the exact solution. Table 4 shows the absolute relative error obtained for three various numbers of terms and at some test points.

Table 4: *Numerical comparison of results in example 4.*

x	*Abs. rel. err.*, (5 Terms)	*Abs. rel. err.*,(10 Terms)	*Abs. rel. err.*, (15 Terms)
0.2	9.10218E-10	0	0
0.4	9.40244E-07	7.01122E-14	6.49916E-15
0.6	5.50308E-05	5.3086E-10	4.48908E-12
0.8	0.000998396	3.04049E-07	2.23E-10

Example 5. Let us consider the nonlinear Fredholm integro-differential equation (Chang and Chang, 2008)

$$(x^3+1)y^{(4)}(x)-8y'(x)/315=x^2/8+\int_0^1 xt(x+t)y^3(t)dt,\ 0\le x\le 1, \tag{19}$$

with the initial conditions

$$y(0)=-1,\ y'(0)=0,\ y''(0)=2 \text{ and } y'''(0)=0. \tag{20}$$

The differential transformation of this equation and its initial conditions are

$$Y(k+4)=\frac{k!}{(k+4)!}\left[-\frac{k!}{(k-3)!}Y(k)+\frac{8(k+1)}{315}Y(k+1)+\left(\frac{1}{8}+\alpha\right)\delta(k-2)+\beta\delta(k-1)\right], \tag{21}$$

where

$$\alpha=\int_0^1 ty^3(t)dt, \text{ and } \beta=\int_0^1 t^2y^3(t)dt, \tag{22}$$

and

$$Y(0)=-1,\ Y(1)=0,\ Y(2)=1 \text{ and } Y(3)=0. \tag{23}$$

To obtain $Y(4)$, put $x=0$ into equation (19) and utilizing the transformation (1), hence

$$Y(4)=0. \tag{24}$$

Substitute $k=1$ and 2 into equation (21), one can get the following relations

$$Y(5)=2/4725+\beta/120, \tag{25}$$

$$Y(6)=1/2880+\alpha/360. \tag{26}$$

The following recurrence relation can be obtained from equation (21)

$$Y(k+4)=\frac{k!}{(k+4)!}\left[-\frac{k!}{(k-3)!}Y(k)+\frac{8(k+1)}{315}Y(k+1)\right],\qquad k\ge 3.$$

Utilizing relation (4) for (22), it can be shown that the following equalities hold for α and β

$$\alpha=\sum_{k=2}^{N}A_{k-2}/k \text{ ,and } \beta=\sum_{k=3}^{N}A_{k-3}/k \tag{27}$$

where N is a suitably large integer that represents the number of terms to be chosen, and A_k are obtained from the Adomian polynomials for the nonlinear function $g(y)=y^3$ as follow

$A_0=Y^3(0)=-1$, $A_1=-3Y(1)$, $A_2=3Y(2)+Y(1)Y(2)-3Y^2(1)$,

$A_3=3Y(3)-6Y(1)Y(2))+Y^3(1)$,

$A_4 = 3Y(4) - 6Y(1)Y(3) - 3Y^2(2) + 3Y^2(1)Y(2)$,

$A_5 = 3Y(5) - 6\big(Y(2)Y(3) + Y(1)Y(4)\big) + 3(Y^2(1)Y(3)$

$+ Y(1)Y^2(2))$, $A_6 = 3Y(6) - (3Y^2(3) + 6Y(2)Y(4) + 6Y(1)Y(5)) + Y^3(2) + 6Y(1)Y(2)Y(3)$

$+ 3Y^2(1)Y(4)$.

Taking into account these relations for A_k, one can solve equations in (27) by taking $N = 8$ terms to obtain $\alpha = -1/8$, and $\beta = -16/315$. Substituting these values of α and β into equations (24) and (25), hence missing coefficients $Y(k)$ of the expansion for the unknown function $y(x)$ can be obtained, that is $y(x) = -1 + x^2$, which is the exact solution.

Example 6. Lastly, the following nonlinear system of Volterra integro-differential equations is considered (Arikoglu and Ozkol, 2008; Biazar*et al*., 2009)

$$\begin{aligned} u''(x) &= 1 - x^3/3 - v'^2(x)/2 + \int_0^x [(u^2(t) + v^2(t))/2]dt, \\ v''(x) &= -1 + x^2 - xu(x) + \int_0^x [(u^2(t) - v^2(t))/4]dt, \end{aligned} \tag{28}$$

with the initial conditions

$$u(0) = 1, \quad u'(0) = 2, \quad v(0) = -1, \text{ and } v'(0) = 0. \tag{29}$$

The differential transformation of the system (28) is the following recurrence relations

$$\begin{aligned} U(k+2) &= k!\big(\delta(k) - \delta(k-3)/3 - A1_k/2 + (A2_{k-1} + A3_{k-1})/2k\big)/(k+2)!, \\ V(k+2) &= k!\big(-\delta(k) - \delta(k-2) - U(k-1) + (A2_{k-1} - A3_{k-1})/4k\big)/(k+2)!, \end{aligned} \tag{30}$$

where $A1_k$, $A2_k$ and $A3_k$ are the differential transforms of the nonlinear functions $g_1(v') = v'^2$, $g_2(u) = u^2$ and $g_3(v) = v^2$, respectively. The initial conditions in equations (29) are transformed by using equation (1) as follows

$$U(0) = 1, \quad U(1) = 2, \quad V(0) = -1 \text{ and } V(1) = 0. \tag{31}$$

Utilizing the lemma, we can get

$A1_0 = V^2(1)$, $A1_1 = 2V(2)V^2(1)$, $A1_2 = 3V(3)V^2(1) + 4V^2(2)$,

$A1_3 = 4V(4)V^2(1) + 12V(2)V(3)$,

$A1_4 = 5V(5)V^2(1) + 16V(2)V(4) + 9V^2(3)$ $A1_5 = 6V(6)V^2(1) + 24V(3)V(4) + 20V(2)V(5)$,

$A1_6 = 7V(7)V^2(1) + 16V^2(4) + 30V(3)V(5) + 24V(2)V(6)$, ...,

$A2_0 = U^2(0)$, $A2_1 = 2U(1)U(0)$, $A2_2 = 2U(2)U(0) + U^2(1)$,

$A2_3 = 2U(3)U(0) + 2U(1)U(2)$,

$A2_4 = 2U(4)U(0) + 2U(1)U(3) + U^2(2)$ $A2_5 = 2U(5)U(0) + 2U(2)U(3) + 2U(1)U(4)$,

$A2_6 = 2U(6)U(0) + U^2(3) + 2U(2)U(4) + 2U(1)U(5)$, ...,

$A3_0 = V^2(0)$, $A3_1 = 2V(1)V(0)$, $A3_2 = 2V(2)V(0) + V^2(1)$, $A3_3 = 2V(3)V(0) + 2V(1)V(2)$,

$A3_4 = 2V(4)V(0) + 2V(1)V(3) + V^2(2)$, $A3_5 = 2V(5)V(0) + 2V(2)V(3) + 2V(1)V(4)$,

$A3_6 = 2V(6)V(0) + V^2(3) + 2V(2)V(4) + 2V(1)V(5)$,... .

By using recurrence relations in system (30), the transformed initial conditions in (31), and last relations for $A1_k$, $A2_k$ and $A3_k$, one can easily evaluate $U(k)$ and $V(k)$. Hence, utilizing the inverse rule in (2) the series solution of the system (28) and (29) up to $O(x^{10})$ is given by

$$u(x)=1+2x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+\frac{x^8}{40320}+\frac{x^9}{362880}+O(x^{10}),$$

$$v(x)=-1-\frac{x^2}{2}-\frac{x^3}{6}-\frac{x^4}{24}-\frac{x^5}{120}-\frac{x^6}{720}-\frac{x^7}{5040}-\frac{x^8}{40320}-\frac{x^9}{362880}-O(x^{10}).$$

For sufficiently large number of terms, the closed form of the solution is $u(x)=x+e^x$, $v(x)=x-e^x$ which is the exact solution. Table 5 shows the absolute relative error obtained for three various numbers of terms and at some test points.

Table 5: *Numerical comparison of results in example 6.*

x	*Abs. rel. err.*, (5 Terms)	*Abs. rel. err.*,(10 Terms)	*Abs. rel. err.*, (15 Terms)
0.2	6.43685E-08	0	0
0.4	3.18791E-06	5.74529E-13	0
0.6	2.92308E-05	3.94906E-11	0
0.8	0.000135599	7.61776E-10	0

Conclusion:

In this work, we present a new approach for applying the differential transform method for solving nonlinear integro-differential equations. The differential transform of the nonlinear term is replaced in the recurrence relation by its Adomian polynomial of index k. Hence, the dependent variable components are replaced by their corresponding differential transforms of the same index. The considered prototype examples include Volterra, Fredholm and coupled system of integro-differential equations with different types of nonlinearity. These numerical examples have proved good results. Here, all algebraic computations are executed by using the *Mathematica* software package.

REFERENCES

Abbasbandy, S., A. Taati, 2009. Numerical solution of the system of nonlinear Volterra integro-differential equations with nonlinear differential part by the operational Tau method and error estimation, J. Comput. Appl. Math, 231: 106-113.

Araghi, M.A.F., Sh. S. Behzadi, 2009. Solving nonlinear Volterra-Fredholm integro-differential equations using the modified Adomian decomposition method, Comput. Meth. Appl. Math, 9: 321-331.

Arikoglu, A., I. Ozkol, 2005. Solution of boundary value problems for integro-differential equations by using differential transform method, Appl. Math. Comput. 168: 1145-1158.

Arikoglu, A., I. Ozkol, 2008. Solution of integral and integro-differential equation systems by using differential transform method, Comput. Math. Appl., 65: 2411-2417.

Aziz, I., S. Islam, 2013. New algorithms for the numerical solution of nonlinear Fredholm and Volterra integral equations using Haar wavelets, J.Comput. and Appl. Math., 239: 333-345.

Biazar, J., H. Ghazvini, M. Eslami, 2009. He's homotopy perturbation method for systems of integro-differential equations, Chaos, Solitons and Fractals, 39: 1253-1258.

Biazar, J., M. Eslami, 2011. Differential transform method for systems of Volterra integral equations of the second kind and comparison with homotopy perturbation method, Int. J. Phys. Sci., 6: 1207-1212.

Borhanifar, A., R. Abazari, 2012. Differential transform method for a class of nonlinear integro-differential equations with derivative type kernel, Canad. J. Comput. Math., Natural Sciences, Eng. and Mid., 3: 1-6.

Chang, S.H., I.L. Chang, 2008. A new algorithm for calculating one-dimensional differential transform of nonlinear functions, Appl. Math. Comput., 195: 977-808.

Darania, P., E. Abadian, A.V. Oskoi, 2006. Linearization method for solving nonlinear integral equations, Math. Probl. Eng.1–10 (doi: 10.1155/MPE/2006/73714).

Darania, P., K. Ivaz, 2008. Numerical solution of nonlinear Volterra-Fredholm integro-differential equations, Appl. Math. Comput., 56: 2197-2209.

Duan, J.S., 2010. Recurrence triangle for Adomian polynomials, Appl. Math. Comput., 216: 1235-1241.

Duan, J.S., 2011. Convenient analytic recurrence algorithm for Adomian polynomials, Appl. Math. Comput., 218: 6337-6348.

Ebadi, G., M.Y. Rahimi-Ardabili, S. Shahmorad, 2007. Numerical solution of the nonlinear Volterra integro-differential equations by the Tau method, Appl. Math. Comput., 188: 1580-1586.

Elsaid, A., 2012. Fractional differential transform method combined with Adomian polynomials, Appl. Math. Comput., 218: 6899-6911.

Kythe, P.K., P. Puri, 1992. Computational Methods for Linear Integral Equations, University of New Orleans, New Orleans.

Maleknejad, K., Y. Mohmoudi, 2003. Taylor Polynomial solution of high-order nonlinear Volterra-Fredholm integro-differential equations, Appl. Math. Comput., 145: 641-653.

Maleknejed, K., F. Mirzaee, 2006. Numerical solution of integro-differential equations by using rationalized Haar functions method, Kybernetes Int. J. Syst. Math., 35: 1735-1744.

Maleknejad, K., B. Basirat, E. Hashemizadeh, 2011. Hybrid Legendre polynomials and Block-pulse functions approach for nonlinear Volterra-Fredholm integro-differential equations, Comput. Math. Appl., 61: 2821-2828.

Odibat, Z.M., 2008. Differential transform method for solving Volterra integral equation with separable kernels, Math. Comput. Model., 48: 1144-1149.

Rashed, M.T., 2004. Numerical solution of functional differential, integral and integro-differential equations, Appl. Numer. Math., 156: 485-492.

Reihani, M.H., Z. Abadi, 2007. Rationalized Haar functions method for solving Fredholm and Volterra integral equations, J. Comput. Appl. Math., 200: 12-20.

Wazwaz, A.M., 2006. A comparison study between the modified decomposition method and traditional method, Appl. Math. Comput., 181: 1703-1712.

Wazwaz, A.M., 2010. The combined Laplace transform-Adomian decomposition method for handling nonlinear Volterra integro-differential equations, Appl. Math. Comput., 216: 1304-1309.

Yalcinbas, S., 2012. Taylor polynomial solution of nonlinear Volterra-Fredholm integral equations, Appl. Math. Comput., 127: 195-206.

Zhou, J.K., 1986. Differential Transformation and Its Applications for Electrical Circuits, Huazhong University Press, Wuhan, China.

Zhao, J., R.M. Corless, 2006. Compact finite difference method for integro-differential equations, Appl. Math. Comput., 177: 271-288.