Australian Journal of Basic and Applied Sciences, 7(7): 364-369, 2013
ISSN 1991-8178

Existence and Uniqueness of Solutions to Impulsive Fractional Integro-Differential Equations

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Abstract; In this paper we investigate the existence and uniqueness of solutions to impulsive fractional integro – differential equations involving the caputo fractional derivative by using Banach and Schaefer's fixed point methods, also this investigation leads us to extent some results gained by BENCHOHRA (Benchohra, M., and Slimani, B. A., 2009)

Key words: Fractional derivative; Impulsive conditions; Existence and uniqueness; Initial value problem; fixed point.

INTRODUCTION

This paper studies the existence and uniqueness of solutions for the initial value problem (IVP for short), for fractional order integro – differential equations.

$$^{C}D^{\alpha}\, x(t) = f(t,x) + \int_0^t k(t,s,x(s))\,ds\,,\; t \in [0,T],\; t \neq t_k \qquad (1.1)$$

$$\Delta x\,|_{t=t_k} = I_k\,(x(t_k^-)) \qquad (1.2)$$

$$x(0) = x_0 \qquad (1.3)$$

Where k= 1,2, ….., m, $0 < \alpha \leq 1,\; J = [0,T],\; f : JxR \longrightarrow R$ *is a* given function, $I_k : R \to R$, *and* $x_0 \in R,\, 0 = t_0 < t_1 < t_2 < .... < t_m < t_{m+1} = T$

$\Delta x\,|_{t=t_k} = x(t_k^+) - x(t_k^-),\; x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ *and* $x(t_k^-) = \lim_{h\to 0^-} x(t_k + h)$

represent the right and left lim *its of* $x(t)$ *at* $t = t_k$

Differential equations of fractional order are extensively applied to many fields of science and engineering, we can numerous applications in Polymer physics (Friedrich, C., 1993; Köstner, M, S. Vlbricht, M., 2011) metrial molding (Magin, R. L., 2010), viscoelasticity (Diethelm, K. and Freed, A.D., 1999), and biology (Pritz, T., 2003). On the other hand , the theory of the impulsive differential equations is also an important area of research which has been investigated in the last few years as mathematical models of phenomena in both physical and social sciences. There has been a significant development in impulsive theory especial in the area of impulsive differential equations with fixed moments: see for instance the monographs by (Bainov, D.D. and Simeonov, P.S., 1989; Benchohra, M., *et al*, 2006; lakshmikantham, V., *et al*, 1989 and Samoilenko, A.M. and Perestyuk, N.A., 1995).

In this paper, we present the existence for problem (1.1) – (1.3) by the application of the contraction principle and Schaefer's fixed point theorem.

Preliminaries:

In this section, we introduce notation, definitions and preliminary fact which are used throughout this paper. By C(J, $\Re$) we denote the Banach space of all continuous functions from J into $\Re$ with the norm

$$\|x\|_\infty := \sup\{|x(t)| : t \in J\}.$$

Definition 2.1 (Kilbas, A.A., *et al*, 2006; Podlubny, I., 1999). The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\Re_+)$ *of order* $\alpha \in \Re_+$ *is defined by*

$$I_a^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\,ds\,,$$

where Γ is the gamma function.

Definition 2.2 (Kilbas, A.A., *et al*, 2006; Podlubny, I., 1999). For a function h defined on the interval [a,b], the α th Riemann – Liouville fractional – order derivative of h is defined by

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$$(D_{a^+}^{\alpha} h)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)\,ds.$$

Here n=[α]+1 and [α] denotes the integer part of α .

Definition 2.3 (Kilbas, A.A., and Marzan, S.A., 2005). For a function h, defined on the interval [a,b], the Caputo fractional – order derivative of order α of h is defined by

$$({}^C D_{a^+}^{\alpha} h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)\,ds.$$

Where n= [α]+1.

Existence Of Solutions:

Let us start by defining what we mean by a solution of the problem (1.1) – (1.3).

Consider the set of the functions

$$\text{PC(J, R)} = \{x : J \to \Re : x \in C((t_k, t_{k+1}], \Re),\ k = 0,1,.....,\ m \text{ and there exist } x(t_k^-) \text{ and } x(t_k^+),$$
$$k = 1,2,....m, \text{ with } x(t_k^-) = x(t_k^+)\}.$$

This set is a Banach space with norm

$$\| x \|_{PC} := \sup_{t\in J} | x(t) |$$

Let J' := [0,T] \ {t_1,…….,t_m}.

Definition 3.1 A function x∈ PC(J,$\Re$) whose $\alpha -$ derivative exists on J' is said to be a solution of (1.1) – (1.3) if x satisfies the equation.

$${}^C D^{\alpha} x(t) = f(t, x(t)) + \int_a^t k(t,s,x(s))\,ds \quad \text{on J' and satisfy the condition}$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)),\ k = 1,.....,m$$

x(0)=x_0

To prove the existence of solutions to (1.1) – (1.3), we need the following auxiliary lemmas.

Lemma 3.1 (Zhang, S., 2006). Let $\alpha > 0$, then the differential equation

$${}^C D^{\alpha} h(t) = 0$$

has a solution h(t)= $c_0+c_1t+c_2t^2+…..+ c_{n-1}t^{n-1}$, $c_i \in \Re$, i=0,1,…..,n-1, n=[α]+1.

Lemma 3.2 (Zhang, S., 2006) Let α >0 , then

$$I^{\alpha\, C} D^{\alpha} h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots.. + c_{n-1} t^{n-1}$$

for some $c_i \in$ R, i=0,1,…..,n-1, n=[α]+1.

As a consequence of lemma 3.1 and 3.2 we have the following result which is useful in what follows.

Lemma 3.3. let $0<\alpha\le 1$ and let $h: J \to \Re$ be continuous A function x is a solution of the fractional integral equation

$$x(t) = \begin{cases} x_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} h(s)\,ds + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} h(s)\int_0^s k(s,\sigma,x(\sigma))\,d\sigma ds \ \ if\ \ t\in[0,t_1] \\ x_0 + \dfrac{1}{\Gamma(\alpha)}\left[\displaystyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}h(s)\,ds + \int_{t_k}^{t}(t-s)^{\alpha-1}h(s)\,ds + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\int_0^s k(s,\sigma,x(\sigma))\,d\sigma ds \right. \\ \left. + \displaystyle\int_{t_k}^{t}(t-s)^{\alpha-1}\int_0^s k(s,\sigma,x(\sigma))\,d\sigma dx \right] + \displaystyle\sum_{i=1}^{k} I_i(x(t_i^-)) \ \ if\ \ t\in(t_k,t_{k+1}]. \end{cases} \quad (3.1)$$

where k=1, …… ,m, if and only if x is a solution of the fractional IVP

$${}^C D^{\alpha}\ x(t) = h(t) + \int_0^t k(t,s,x(s))\,ds \quad (3.2)$$

$$\Delta x|_{t=t_k} = I_k\ (x(t_k^-))\ , k = 1,......,\ \ m \quad (3.3)$$

$$x(0) = x_0 \quad (3.4)$$

for proof see (M. Benchohra and B. A. Slimani, 2009). Our first result is based on Banach fixed point theorem.

Theorem 3.1 Assume that

(H1) There exists a constant L>0 such that

$$\left| f(t,z) - (f(t,\bar{z})\right| \le L\left| z - \bar{z}\right|, \; for \; some \; t \in J, and \; each \; z, \bar{z} \in \Re.$$

(H2) There exists a constant Q>0 such that

$$\left| I_k(z) - I_k(\bar{z})\right| \le Q\left| z - \bar{z}\right|, \; for \; some \; z, \bar{z} \in \Re.$$

(H3) There exists a constant P>0 such that

$\left| k(t,s,z(s)) - k(t,s,\bar{z}(s))\right| \le P \mid z - \bar{z}|,$ If

$$\left[\frac{(m+1)LT^{\alpha}}{\Gamma(\alpha+1)} + \frac{(m+1)PT^{\alpha+1}}{\Gamma(\alpha+2)} + mQ \right] < 1 \qquad (3.5)$$

then (1.1) – (1.3) has a unique solution on J.

Proof: Consider the operator $F : PC(J, \Re) \to PC(J, \Re)$ defined by

$$F(x)(t) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, x(s)) ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, x(s)) ds$$

$$\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \int_0^s k(s,\sigma,x(\sigma)) d\sigma ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \int_0^s k(s,\sigma,x(\sigma)) d\sigma ds \qquad (3.6)$$

$$+ \sum_{0<t_k<t} I_k(x(t_k^-))$$

clearly, the fixed points of the operator F are solutions of the problem (1.1) - (1.3). We shall use the Banach contraction Principle to prove that F defined by (3.6) has a fixed point. We claim that F is a contraction mapping. Let x,y $\in$ PC(J, $\Re$). Then, for each $t \in J$ we have

$$\left| F(x)(t) - F(y)(t)\right| \le \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \left| f(s,x(s)) - f(s,y(s))\right| ds +$$

$$\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} \left| f(s,x(s)) - f(s,y(s))\right| dy +$$

$$\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \int_0^s \left| k(s,\sigma,x(\sigma)) - k(s,\sigma,y(\sigma))\right| d\sigma dy +$$

$$\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_k}^{t} (t - s)^{\alpha-1} \int_0^s \left| k(s,\sigma,x(\sigma)) - k(s,\sigma,y(\sigma))\right| d\sigma dy +$$

$$\sum_{0<t_k<t} \left| I_k(x(t_k^-)) - I_k(y(t_k^-))\right|$$

$$\le \frac{1}{\Gamma(\alpha)} \left[\sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} L\left| x(s) - y(s)\right| dy + \int_{t_k}^{t} (t-s)^{\alpha-1} L(\left| x(s) - y(s)\right| ds \right.$$

$$+ \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \int_0^s P\left| x(s) - y(s)\right| d\sigma dy + \int_{t_k}^{t_k} (t - s)^{\alpha-1} \int_0^s P\left| x(s) - y(s)\right| d\sigma ds$$

$$+ \sum_{k=1}^{m} Q\left| x(t_k^-) - y(t_k^-)\right|.$$

$$\le \left[\frac{(m+1)\Gamma T^{\alpha}}{\Gamma(\alpha+1)} + \frac{(m+1)PT^{\alpha+1}}{\Gamma(\alpha+2)} + mQ \right] \left\| x - y\right\|_{\infty}$$

Consequently by (3.5), F is a contraction. As a consequence of Banach fixed point theorem, we deduce that F has a fixed point whish is a solution of the problem (1.1)-(1.3)

Theorem 3.2 Assume that

(H4) The function $f : J x \Re \to \Re$ is continuous.

(H5) There exists a constant M>0 such that $|f(t,z)| \le M$ *for each* $t \in J$ *and* $z \in \Re$.

(H6) Let $k : [0,T] x [0,T] x \Re \to \Re$ be continuous and there exists a constant M_1>0 such that $|k(t,s,z(s))| \le M_1$ *for each* $z \in \Re, t \in J$.

(H7) The functions $I_k : \Re \to \Re$ are continuous and there exists a constant M_2>0, such that $|I_k(z)| \le M_2$ *for each* $z \in \Re, k = 1,...., m$.

Then (1.1)-(1.3) has at least one solution on J.

Proof: We shall use Schaefer's fixed point theorem to prove that F defined by (3.6) has a fixed point, and so we claim that F is continuous.

Let {x_n} be a sequence such that $x_n \to x$ *in* $PC(J,\Re)$, *Then for each* $t \in J$

$$\|F(x_n)(t) - F(x)(t)\| \le \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |f(s,x_n(s)) - f(s,x(s))| ds +$$

$$\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \int_0^s |k(s,\sigma,x(\sigma)) - k(s,\sigma,y(\sigma))| d\sigma dy +$$

$$\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |f(s,x_n(s)) - f(s,x(s))| ds +$$

$$\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} \int_0^s |k(s,\sigma,x(\sigma)) - k(s,\sigma,y(\sigma))| d\sigma dy$$

$$+ \sum_{0<t_k<t} |I_k(x_n(t_k^-)) - I_k(x(t_k^-))|$$

Since f, k and I_k, k=1,2, …, m are continuous functions, and $x_n \to x$ *we have*

$$\|F(x_n) - F(x)\|_\infty \to 0 \text{ as } n \to \infty.$$

Thus, F is continuous, Next, we prove that F maps bounded sets into bounded sets in PC(J, $\Re$). Indeed, it is enough to show that for any r>0 there exists a positive constant l such that for each

$x \in B_r = \{x \in PC(J,\Re) : \|x\|_\infty \le r\}$, *we have* $\|F(x)\|_\infty \le l$.

By (H5) and (H7), we have

$$|F(x)(t)| \le |x_0| + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |f(s,x(s))| ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |f(s,x(s))| ds +$$

$$\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \int_0^s |k(s,\sigma,x(\sigma))| d\sigma ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} \int_0^s |k(s,\sigma,x(\sigma))| d\sigma ds +$$

$$\sum_{0<t_k<t} |I_k(x(t_k^-))|$$

$$\le |x_0| + \frac{MT^{\alpha}(m+1)}{\Gamma(\alpha+1)} + \frac{M_1 T^{\alpha+1}(m+1)}{\Gamma(\alpha+2)} + mM_2$$

Thus $\|F(x)\|_\infty \le |x_0| + \frac{MT^{\alpha}(m+1)}{\Gamma(\alpha+1)} + \frac{M_1 T^{\alpha+1}(m+1)}{\Gamma(\alpha+2)} + mM_2 := l.$

Now, we claim that F maps bounded sets into equicontinuous sets of PC(J, $\Re$). Let $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$, B_r be bounded set of PC(J, $\Re$), and let $x \in B_r$, then

$$\left|F(x)(\tau_2)-F(x)(\tau_1)\right| \le +\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\left[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right]f(s,x(s)|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}\left|f(s,x(s)\right|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\left[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right]\int_0^{s}\left|k(s,\sigma,x(\sigma))\right|d\sigma ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\tau}^{\tau_2}(\tau_2-s)^{\alpha-1}\int_0^{s}\left|k(s,\sigma,x(\sigma))\right|d\sigma ds+\sum_{0<t_k<t}\left|I_k(x(t_k^-))\right|$$
$$\le \frac{M}{\Gamma(\alpha+1)}\left[2(\tau_2-\tau_1)^{\alpha}-(\tau_2^{\alpha}-\tau_1^{\alpha})\right]+\frac{M_1}{\Gamma(\alpha+2)}\left[2(\tau_2-\tau_1)^{\alpha+1}-(\tau_2^{\alpha+1}-\tau_1^{\alpha+1})\right]$$
$$+\frac{M_1}{\Gamma(\alpha+1)}\left[2\tau_1(\tau_2-\tau_1)^{\alpha}\right]+\sum_{0<t_k<\tau_2-\tau_1}\left|I_k(x(t_k^-))\right|$$

As $\tau_1 \to \tau_2$, the right – hand side of the above inequality tends to zero. As a consequence of Arzela-Ascoli theorem, we can conclude that.

$F: PC(J,\Re) \to PC(J,\Re)$ is completely continuous. Now it remains to show that the set $\varepsilon = \{x \in PC(J,\Re): x = \lambda F(x) \text{ for some } 0<\lambda<1\}$ is bounded.
Let $x \in \varepsilon$, then $x=\lambda F(x)$ for some $0<\lambda<1$. Thus for $t \in J$ we have

$$x(t)=\lambda x_0+\frac{\lambda}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,x(s))ds$$
$$+\frac{\lambda}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\int_0^{s}k(s,\sigma,x(\sigma))d\sigma ds+\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,x(s))ds$$
$$+\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\int_0^{s}k(s,\sigma,x(\sigma))d\sigma ds+\lambda\sum_{0<t_k<t}\left|I_k(x(t_k^-))\right|$$

This implies by (H5), (H6) and (H7) that for each $t \in J$ we have

$$|x(t)| \le |x_0|+\frac{MT^{\alpha}(m+1)}{\Gamma(\alpha+1)}+\frac{M_1T^{\alpha+1}(m+1)}{\Gamma(\alpha+2)}+mM_2$$ Thus for every $t \in J$, we have

$$\|F(x)\|_{\infty} \le |x_0|+\frac{MT^{\alpha}(m+1)}{\Gamma(\alpha+1)}+\frac{M_1T^{\alpha+1}(m+1)}{\Gamma(\alpha+2)}+mM_2 := l^*.$$

Hence the set ε is bounded. As a consequence of Schafer's fixed point theorem, we have that F has a fixed point which is a solution of the problem (1.1) – (1.3)

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