Australian Journal of Basic and Applied Sciences, 7(7): 658-667, 2013
ISSN 1991-8178

Transmuted Exponentiated Lomax Distribution

S.K. Ashour and M.A. Eltehiwy

Institute of Statistical Studies & Research, Cairo University, Egypt.

Abstract: A generalization of the exponentiated Lomax distribution so-called the transmuted exponentiated Lomax distribution is proposed and studied. Various structural properties including explicit expressions for the moments, quantiles, and mean deviations of the new distribution are derived. The estimation of the model parameters is performed by maximum likelihood method. We hope that the new distribution proposed here will serve as an alternative model to the other models which are available in the literature for modeling positive real data in many areas.

Key words: Exponentiated Lomax distribution, hazard rate function, reliability function, parameter estimation

INTRODUCTION

The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of this, considerable effort has been expended in the development of large classes of standard probability distributions along with revelent statistical methodologies. In fact, the statistics literature is filled with hundreds of continuous univariate distributions. However, in recent years, applications from the environmental, financial, biomedical sciences, engineering among others, have further shown that data sets following the classical distributions are more often the exception rather than the reality. Since there is a clear need for extended forms of these distributions a significant progress has been made toward the generalization of some well-known distributions and their successful application to problems in areas such as engineering, finance, economics and biomedical sciences, among others.

In this article we use transmutation map approach suggested by Shaw and Buckley (2007) to define a new model which generalizes the Exponentiated Lomax model. We will call the generalized distribution as the transmuted Exponentiated Lomax (TEL) distribution. According to the Quadratic Rank Transmutation Map, (QRTM), approach the cumulative distribution function (cdf) satisfy the relationship

$$F_2(x) = (1+\lambda)F_1(x) - \lambda F_1(x)^2 \tag{1}$$

which on differentiation yields,

$$f_2(x) = f_1(x)[1+\lambda - 2\lambda F_1(x)] \tag{2}$$

where $f_1(x)$ and $f_2(x)$ are the corresponding probability density function (pdf) associated with $F_1(x)$ and $F_2(x)$ respectively and $-1 \leq \lambda \leq 1$. An extensive information about the quadratic rank transmutation map is given in Shaw and Buckley (2007).

We will use the above formulation for a pair of distributions $F(x)$ and $G(x)$ where $G(x)$ is a submodel of $F(x)$. therefore, a random variable X is said to have a transmuted probability distribution with cdf $F(x)$ if

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2, \qquad |\lambda| \leq 1 \tag{3}$$

where $G(x)$ is the cdf of the base distribution. Observe that at $\lambda = 0$ we have the distribution of the base random variable. Aryal and Tsokos (2009, 2011) studied the transmuted extreme distributions. The authors provided the mathematical characterization of transmuted Gumbel and transmuted Weibull distributions and their applications to analyze real data sets. Aryal (2013) studied the transmuted log logistic distribution and discussed some properties of this family. In the present study we will provide mathematical formulation of the transmuted exponentiated Lomax (TEL) distribution and some of its properties.

Transmuted Exponentiated Lomax Distribution:

Abdul-Moniem and Abdel-Hameed (2012) generalized the Lomax distribution by powering a positive real number (α) to the cumulative distribution function (cdf). This new family of distributions called exponentiated Lomax distribution.

Corresponding Author: S.K. Ashour, Institute of Statistical Studies & Research, Cairo University, Egypt.
E-mail: ashoursamir@hotmail.com

A random variable X is said to have a exponentiated Lomax distribution with parameters θ, α and $\gamma > 0$ if its probabability density function (pdf) is given by

$$g(x) = \alpha\theta\gamma[1-(1+\gamma x)^{-\theta}]^{\alpha-1}(1+\gamma x)^{-(\theta+1)} \quad x > 0, \theta, \alpha \text{ and } \gamma > 0 \tag{4}$$

and the cdf of X is given by

$$G(x) = \left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}. \quad x > 0, \quad \theta, \alpha \text{ and } \gamma > 0 \tag{5}$$

Now using (3) and (5) we have the cdf of a transmuted exponentiated Lomax

$$F(x) = \left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\left((1+\lambda) - \lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right) \tag{6}$$

Hence, the pdf of the transmuted exponentiated Lomax distribution with parameters θ, α, γ and λ is

$$f(x) = \frac{\alpha\theta\gamma\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha-1}}{(1+\gamma x)^{(\theta+1)}}\left(1+\lambda-2\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right). \tag{7}$$

Note for $\lambda = 0$ and $\alpha = 1$, we have the pdf of exponentiated Lomax distribution and transmuted Lomax distribution respectively. Figure 1 illustrates some of the possible shapes of the density function of transmuted exponentiated Lomax distribution for selected values of the parameters.

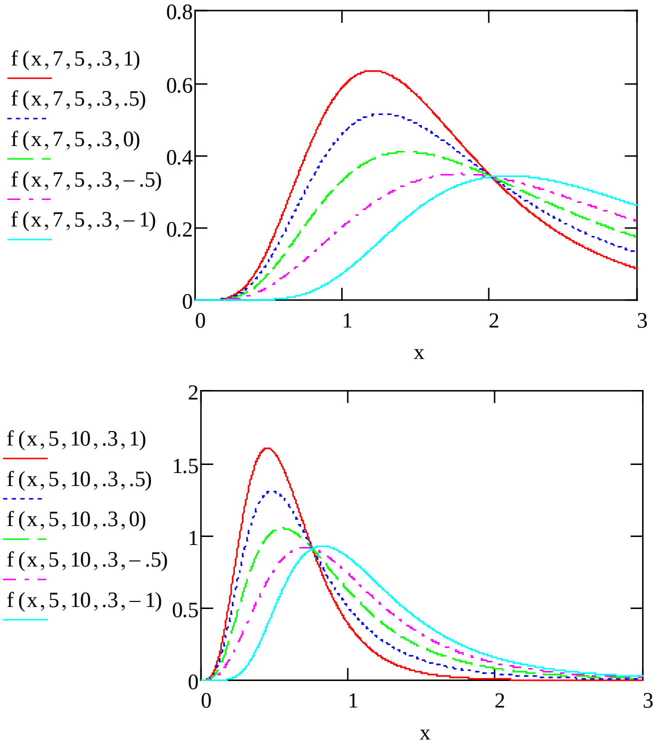


Fig. 1: The pdf of TEL distrbution for $\alpha = 7, 5, \;\; \theta = 5, 10, \;\; \gamma = 0.3$ when $\lambda = -1, -0.5, 0, 0.5, 1$.

Moments and Quantiles:

In this section we shall present the moments and quantiles for the transmuted exponentiated Lomax distribution. The r^{th} moments of a transmuted exponentiated Lomax random variable X, is given by

$$E(X^r) = \int_0^{\infty} \frac{\alpha\theta\gamma x^r\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha-1}}{(1+\gamma x)^{(\theta+1)}}\left(1+\lambda-2\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right),$$

$$= \frac{\alpha(1+\lambda)}{\gamma^r}\sum_{i=0}^{r}\binom{r}{i}(-1)^i B\left(1-\frac{1}{\theta}(r-i), \alpha\right) - \frac{2\alpha\lambda}{\gamma^r}\sum_{i=0}^{r}\binom{r}{i}(-1)^i B\left(1-\frac{1}{\theta}(r-i), 2\alpha\right), \tag{8}$$

where, $B(.,.)$ is the beta function defined by
$B(\alpha,b)=\int_0^1 t^{\alpha-1}(1-t)^{b-1}dt.$

In particular, the mean of the TEL distribution is given by

$$E(X)=\frac{\alpha(1+\lambda)}{\gamma}\left[B\left(1-\frac{1}{\theta},\alpha\right)-\frac{1}{\alpha}\right]-\frac{2\alpha\lambda}{\gamma}\left[B\left(1-\frac{1}{\theta},2\alpha\right)-\frac{1}{2\alpha}\right] \quad (9)$$

Table 1 lists the first four ordinary moments for selected values of the parameter λ of the transmuted exponentiated Lomax distribution for $\alpha=5$, γ =0.3 and θ =10.

Using these ordinary moments one can easily compute the variance, skewness and kurtosis of the transmuted exponentiated Lomax for the selected values of the parameters.

Table 1: Moments of TELD for selected values of the parameters.

	$\lambda=-1$	$\lambda=-0.5$	$\lambda=0$	$\lambda=0.5$	$\lambda=1$
$r=1$	1.171	1.029	0.888	0.746	0.604
$r=2$	1.747	1.424	1.102	0.779	0.456
$r=3$	3.389	2.647	1.905	1.164	0.422
$r=4$	8.777	6.701	4.626	2.55	0.474

The q^{th} quantile x_q of the transmuted exponentiated Lomax distribution can be obtaned from (6) as

$$x_q=\frac{\left[1-\left[\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda q}}{2\lambda}\right]^{1/\alpha}\right]^{-1/\theta}-1}{\gamma} \quad (9)$$

Hence, the distribution median is

$$x_{0.5}=\frac{\left[1-\left[\frac{1+\lambda-\sqrt{1+\lambda^2}}{2\lambda}\right]^{1/\alpha}\right]^{-1/\theta}-1}{\gamma} \quad (10)$$

To illustrate the effect of the shape parameter λ on skewness and kurtosis we consider measures based on quantiles. The shortcomings of the classical kurtosis measure are well known. There are many heavy-tailed distributions for which this measure is infinite, so it becomes uninformative. The Bowley's skewness (Kenney and Keeping,1962) is one of the earliest skewness measures defined by the average of the quartiles minus the median, divided by the half the interquartile range, given by

$$\mathcal{B}=\frac{Q_3+Q_1-2Q_2}{Q_3-Q_1}=\frac{Q(3/4)+Q(1/4)-2Q(2/4)}{Q(3/4)-Q(1/4)}.$$

and the Moors kurtosis (Moors, 1998) is based on octiles and is given by

$$\mathcal{M}=\frac{(E_3-E_1)+(E_7-E_5)}{E_6-E_2}=\frac{Q(3/8)-Q(1/8)+Q(7/8)-Q(5/8)}{Q(6/8)-Q(2/8)}.$$

For any distribution symmetrical to 0 the Moors kurtosis reduces to
$\mathcal{M}=\frac{(E_7-E_5)}{E_6}.$
t is easy to calculate that for standand normal distribution $E_1=-E_7=-1.15$, $E_2=-E_6=-0.32$. Therefore, $\mathcal{M}=1.23$. Hence, the centered Moor's coefficient is given by

$$\mathcal{M}=\frac{(E_3-E_1)+(E_7-E_5)}{E_6-E_2}-1.23.$$

Fiqures 2 displays the Bowley ($\mathcal{B}$) and Moors ($\mathcal{M}$) kurtosis as a function of the parmeter $1\geq\lambda>0$ and $-1\leq\lambda<0$ for $\alpha=5$, $\gamma=0.3$ and θ =10. It is evident that both measures depend on the parameter λ.

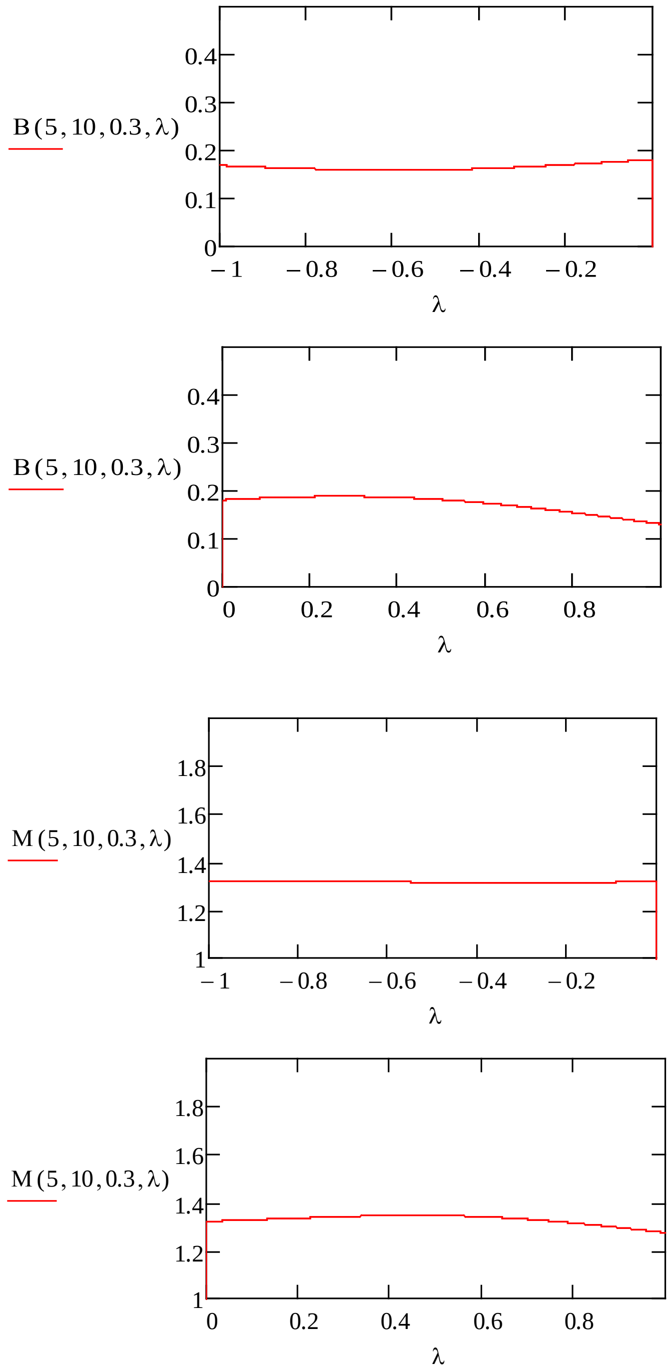


Fig. 2: Behavior of Bowley(B) and Moors(M) kurtosis for TEL distribution.

Mean Deviation:

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and the median. These are known as the mean deviation about the mean and the mean deviation about the median respectively and are defined by

$$\delta_1 = \int_0^{\infty} |x - \mu| f(x)\, dx \tag{11}$$

and

$$\delta_2 = \int_0^{\infty} |x - M| f(x) dx \tag{12}$$

where

$$\mu = E(X) = \frac{\alpha(1+\lambda)}{\gamma}\left[B\left(1-\frac{1}{\theta},\alpha\right)-\frac{1}{\alpha}\right]-\frac{2\alpha\lambda}{\gamma}\left[B\left(1-\frac{1}{\theta},2\alpha\right)-\frac{1}{2\alpha}\right]$$

and

$$M = Median(X) = \frac{\left[1-\left[\frac{1+\lambda-\sqrt{1+\lambda^2}}{2\lambda}\right]^{1/\alpha}\right]^{-1/\theta}-1}{\gamma}.$$

The measures $\delta_1(X)$ and $\delta_2(X)$ can be expressed as $\delta_1(X) = 2\mu F(\mu) - 2J(\mu)$ and $\delta_2(X) = \mu - 2J(M)$ where $J(q) = \int_0^q x\, f(x)dx$. For a transmuted exponentiated Lomax

$$J(q) = (1+\lambda)\alpha\theta\gamma \int_0^q x\,\frac{\alpha\theta\gamma\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha-1}}{(1+\gamma x)^{(\theta+1)}}dx - 2\lambda\alpha\theta\gamma\int_0^q x\,\frac{\alpha\theta\gamma\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}}{(1+\gamma x)^{(\theta+1)}}dx \quad (13)$$

One can easily compute these integrals numerically in software such as Mathcad (Brent, 2006) and hence get the mean deviations about the mean and about the median as desired. From the mean deviations we can construct Lorenz and Bonferroni curves, which are used in several areas including economics, reliability, insurance and medicine and others.

Some numerical values of the mean deviation from mean and median for selected value of for $\alpha = 5$, $\gamma = 0.3$ and θ =10 and different values of λ are listed in the table 2 below.

Table 2: Mean deviation from the mean and the median for selected values of the parameters.

	$\lambda = -1$	$\lambda = -0.5$	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 1$
δ_1	0.445	0.441	0.405	0.334	0.230
δ_2	0.431	0.427	0.888	0.320	0.226

Random Number Generation and Parameter Estimation:

Using the method of inversion we can generate random numbers from the transmuted exponentiated Lomax distribution as

$$\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\left((1+\lambda)-\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right) = u.$$

where $u \sim U(0,1)$. After simple calculation this yields

$$x = \frac{\left[1-\left[\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda u}}{2\lambda}\right]^{1/\alpha}\right]^{-1/\theta}-1}{\gamma} \quad (14)$$

One can use equation (14) to generate random numbers when the parameters α, θ, γ and λ are known. The maximum likelihood estimates, MLEs, of the parameters that are inherent within in the transmuted exponentiated Lomax probability distribution function is given by the following:

Let $X_1, X_2, \ldots X_n$ be a sample size n from a transmuted exponentiated Lomax distribution. Then the likelihood function is given by

$$L = \frac{\alpha^n\theta^n\gamma^n \prod_{i=1}^n\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha-1}\prod_{i=1}^n\left(1+\lambda-2\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right)}{\prod_{i=1}^n(1+\gamma x)^{(\theta+1)}} \quad (15)$$

Hence, the log-likelihood function $\mathcal{L} = \ln L$ becomes

$$\mathcal{L} = n(\ln\alpha + \ln\theta + \ln\gamma) + (\alpha-1)\sum_{i=1}^n \ln\left[1-(1+\gamma x)^{-\theta}\right] - n(\theta+1)\ln(1+\gamma x) + \sum_{i=1}^n \ln\left(1+\lambda-2\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right)$$

Therefore, the MLEs of α, θ, γ and λ which maximize (15) must satisfy the following normal equations

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \ln\left[1-(1+\gamma x)^{-\theta}\right] + \sum_{i=1}^n \frac{-2\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\ln\left(1-(1+\gamma x)^{-\theta}\right)}{\left(1+\lambda-2\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right)},$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{n}{\theta} - (\alpha - 1)\sum_{i=1}^{n}\frac{(1+\gamma x)^{-\theta}\ln(1+\gamma x)}{[1-(1+\gamma x)^{-\theta}]} - n\ln(1+\gamma x) + \sum_{i=1}^{n}\frac{2\lambda\alpha\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha-1}(1+\gamma x)^{-\theta}\ln(1+\gamma x)}{(1+\lambda-2\lambda[1-(1+\gamma x)^{-\theta}]^{\alpha})}$$

$$\frac{\partial \mathcal{L}}{\partial \gamma} = \frac{n}{\gamma} - (\alpha - 1)\sum_{i=1}^{n}\frac{\gamma\theta(1+\gamma x)^{-(\theta+1)}}{[1-(1+\gamma x)^{-\theta}]} - \frac{n(\theta+1)x}{(1+\gamma x)} + \sum_{i=1}^{n}\frac{2\lambda\alpha\gamma\theta\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha-1}(1+\gamma x)^{-(\theta+1)}}{(1+\lambda-2\lambda[1-(1+\gamma x)^{-\theta}]^{\alpha})}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{i=1}^{n}\frac{1-2\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}}{\left(1+\lambda-2\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right)}.$$

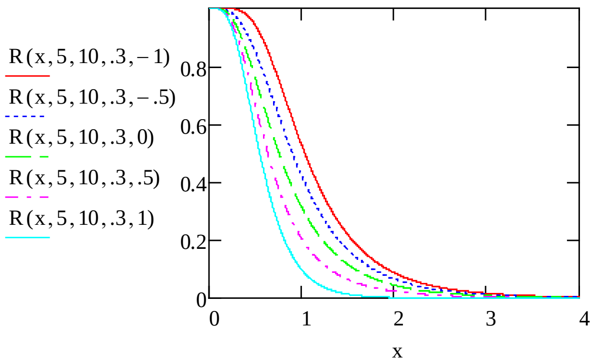
The MLE of α, γ, θ and λ is obtained by solving this nonlinear system of equations. Setting these expressions to zero and solving them simultaneously yields the maximum likelihood estimates of the four parameters.

Reliability Analysis:

The survival function, also known as the reliability function in engineering, is the characteristic of an explanatory variable that maps a set of events, usually associated with mortality or failure of some system onto time. It is the probability that the system will survive beyond a specified time.

The transmuted exponentiated Lomax distribution can be a useful model to characterize failure time of a given system because of the analytical structure. The reliability function $\bar{F}(t)$, which is the probability of an item not failing prior to sometime t, is defined by $\bar{F}(t) = 1 - F(t)$, The reliability function of a transmuted exponentiated Lomax distribution is given by

$$\bar{F}(t) = 1 - \left[1-(1+\gamma t)^{-\theta}\right]^{\alpha}\left((1+\lambda) - \lambda\left[1-(1+\gamma t)^{-\theta}\right]^{\alpha}\right) \tag{16}$$



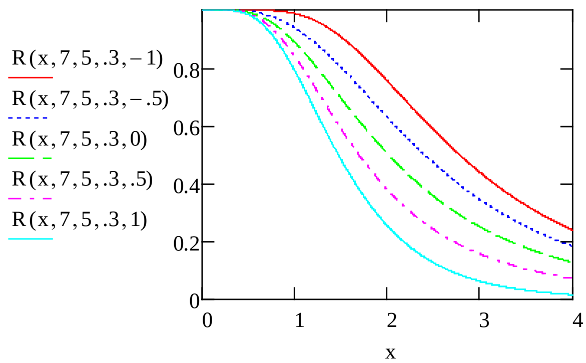


Fig. 3: The reliability function of TEL distrbution for $\alpha = 7, 5$, $\theta = 5, 10$, $\gamma = 0.3$ when $\lambda = -1, -0.5, 0, 0.5, 1$.

The other characteristic of interest of a random variable is the hazard rate function also known as instantaneous failure rate defined by

$$h(t) = \frac{f(t)}{1 - F(t)}$$

which is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the time t.

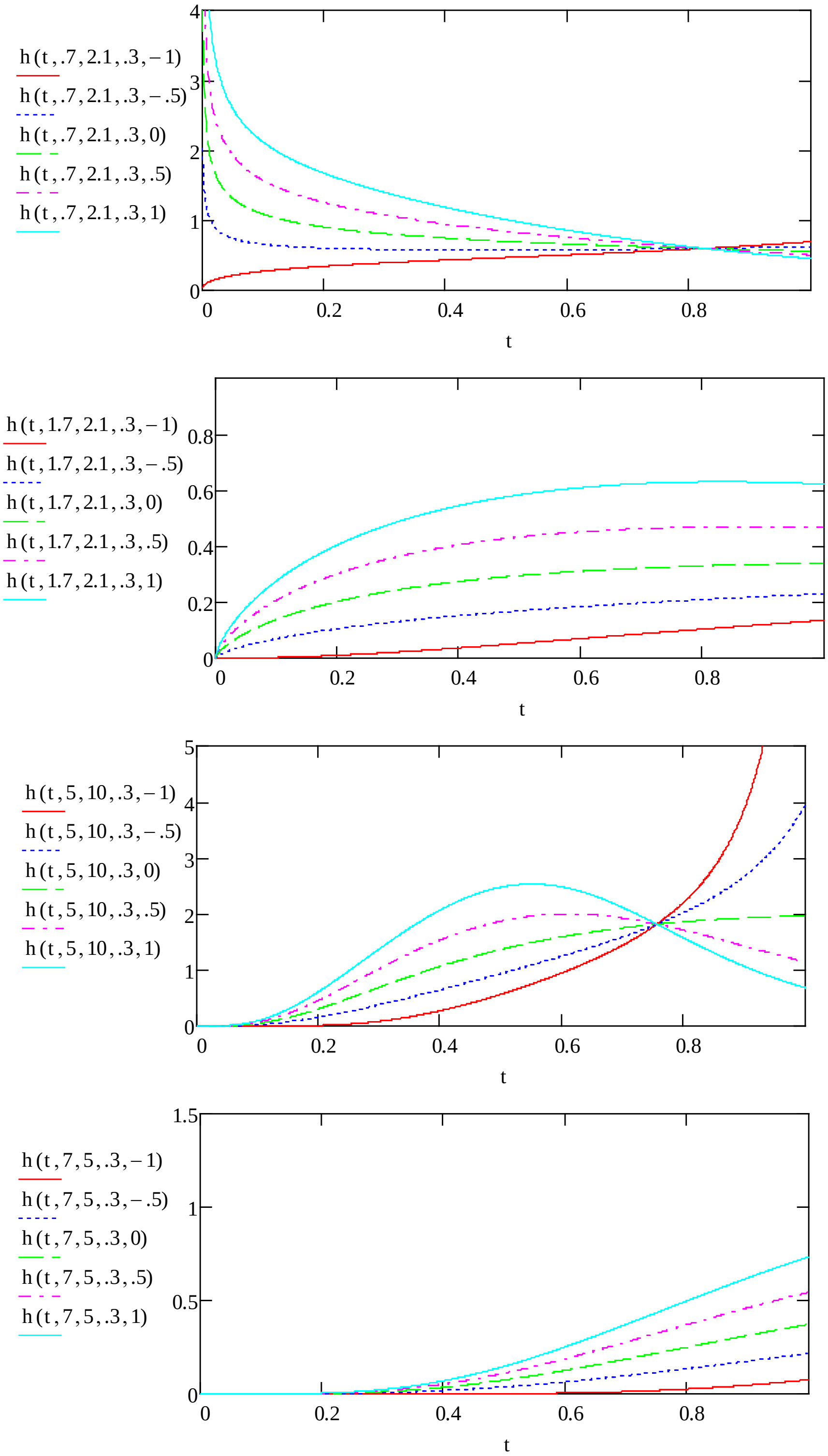


Fig. 4: The Hazard rate function of TEL distrbution for $\alpha = 0.7, 1.7,\ 7,\ 5,\ \ \theta = 2.1,\ 5, 10,\ \gamma = 0.3$ when $\lambda = -1, -0.5, 0, 0.5, 1$.

The hazard rate function for a transmuted exponentiated Lomax distribution is given by

$$h(t) = \frac{\alpha\theta\gamma\left[1-(1+\gamma t)^{-\theta}\right]^{\alpha-1}\left(1+\lambda-2\lambda\left[1-(1+\gamma t)^{-\theta}\right]^{\alpha}\right)}{(1+\gamma t)^{(\theta+1)}\left[1-\left[1-(1+\gamma t)^{-\theta}\right]^{\alpha}\left((1+\lambda)-\lambda\left[1-(1+\gamma t)^{-\theta}\right]^{\alpha}\right)\right]}.$$

It is important to note that the units for $h(x)$ is the probability of failure per unit of time, distance or cycles. Figure 4 illustrates the behavior of the hazard rate function of a transmuted exponentiated Lomax distribution for selected values of the parameters.

Observing the behavior of the hazard rate function it is worth noting that the transmuted exponentiated Lomax distribution will have more applicability than the exponentiated Lomax distribution and some of its generalizations.

The hazard rate function of the transmuted exponentiated Lomax distribution has the following properties:

i. If $\alpha = 1$ the failure rate is same as the transmuted Lomax distribution

$$h(t) = \frac{\theta\gamma\left(1+\lambda-2\lambda\left[1-(1+\gamma t)^{-\theta}\right]\right)}{(1+\gamma t)^{(\theta+1)}[(1+\gamma t)^{-\theta}(1+\lambda-\lambda[1-(1+\gamma t)^{-\theta}])]}$$

ii. If $\lambda = 0$ the failure rate is same as the exponentiated Lomax distribution

$$h(t) = \frac{\alpha\theta\gamma[1-(1+\gamma t)^{-\theta}]^{\alpha-1}}{(1+\gamma t)^{(\theta+1)}[1-[1-(1+\gamma t)^{-\theta}]^{\alpha}]}$$

iii. If If $\lambda = 0, \alpha = 1$ the failure rate is same as the Lomax distribution

$$h(t) = \theta\gamma(1+\gamma t)^{-1}$$

Many generalized probability models have been proposed in reliability literature through the fundamental relationship between the reliability function $\bar{F}(x)$ and its cumulative hazard function (CHF) $H(t)$ given by $-\ln \bar{F}(x)$. The CHF describes how the risk of a particular outcome changes with time. The cumulative hazard rate function of a transmuted exponentiated Lomax distribution is given by

$$H(t) = -\ln\left(1-\left[1-(1+\gamma t)^{-\theta}\right]^{\alpha}\left((1+\lambda)-\lambda\left[1-(1+\gamma t)^{-\theta}\right]^{\alpha}\right)\right)$$

Observe that:

i. $H(t)$ is nondecreasing for all $t \geq 0$,
ii. $H(0) = 0$,
iii. $\lim_{t\to\infty} H(t) = \infty$.

It is important to note that the units for $H(t)$ are the cumulative probability of failure per unit of time, distance or cycles.

Order Statistics:

Order statistics make their appearance in many areas of statistical theory and practice. We know that if $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \dots, X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$, then the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!\,(n-j)!} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{n-j}$$

For $j = 1,2,\dots,n$

We have from (4) and (5) the pdf of the j^{th} order exponentiated Lomax random variable $X_{(j)}$ is given by

$$g_{X_{(j)}}(x) = \frac{n!}{(j-1)!\,(n-j)!}\alpha\theta\gamma(1+\gamma x)^{-(\theta+1)}[1-(1+\gamma x)^{-\theta}]^{(\alpha j-1)}$$
$$\times\{1-[1-(1+\gamma x)^{-\theta}]^{\alpha}\}^{n-j}.$$

Therefore, the pdf of the n^{th} order exponentiated Lomax statistic $X_{(n)}$ is given by

$$g_{X_{(n)}}(x) = n\alpha\theta\gamma(1+\gamma x)^{-(\theta+1)}[1-(1+\gamma x)^{-\theta}]^{(\alpha n-1)} \quad (17)$$

and the pdf of the first order exponentiated Lomax statistic $X_{(1)}$ is given

$$g_{X_{(1)}}(x) = n\alpha\theta\gamma(1+\gamma x)^{-(\theta+1)}[1-(1+\gamma x)^{-\theta}]^{\alpha-1} \times \{1-[1-(1+\gamma x)^{-\theta}]^{\alpha}\}^{n-1} \quad (18)$$

Note that in a particular case of $n = 2$, (17) yields

$$g_{X_{(2)}}(x) = 2\alpha\theta\gamma(1+\gamma x)^{-(\theta+1)}\,[1-(1+\gamma x)^{-\theta}]^{(2\alpha-1)} \quad (19)$$

and (18) yields

$$g_{X_{(1)}}(x) = 2\alpha\theta\gamma(1+\gamma x)^{-(\theta+1)}[1-(1+\gamma x)^{-\theta}]^{\alpha-1}\{1-[1-(1+\gamma x)^{-\theta}]^{\alpha}\} \quad (20)$$

Observe that (19) and (20) are special cases of (7) for $\lambda = -1$ and $\lambda = 1$ respectively. It has been observe that a transmuted exponentiated Lomax distribution with $\lambda = 1$ is the distribution of $min(X_1, X_2)$ and a transmuted exponentiated Lomax with $\lambda = -1$ is the of the $max(X_1, X_2)$ where X_1 and X_2 are independent and identically distributed exponentiated Lomax random variables.

Now we provide the distribution of the order statistics for transmuted exponentiated Lomax random variable. The pdf of the j^{th} order statistic for transmuted exponetiated Lomax distribution is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!\,(n-j)!}\frac{\alpha\theta\gamma[1-(1+\gamma x)^{-\theta}]^{\alpha j-1}}{(1+\gamma x)^{(\theta+1)}} \times \left(1+\lambda-2\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right)$$
$$\times\left((1+\lambda)-\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right)^{(j-1)}$$
$$\times\left\{1-\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\left((1+\lambda)-\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right)\right\}^{n-j}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$f_{X_{(n)}}(x) = n\frac{\alpha\theta\gamma[1-(1+\gamma x)^{-\theta}]^{\alpha n-1}}{(1+\gamma x)^{(\theta+1)}} \times \left(1+\lambda-2\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right)$$
$$\times\left((1+\lambda)-\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right)^{(n-1)}$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$f_{X_{(1)}}(x) = n\frac{\alpha\theta\gamma[1-(1+\gamma x)^{-\theta}]^{(\alpha-1)}}{(1+\gamma x)^{(\theta+1)}} \times \left(1+\lambda-2\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right)$$
$$\times\left\{1-\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\left((1+\lambda)-\lambda\left[1-(1+\gamma x)^{-\theta}\right]^{\alpha}\right)\right\}^{n-1}$$

Note that $\lambda = 0$ yields the order statistics of the exponentiated Lomax distribution.

Concluding Remarks:

In the present study, we have introduced a new generalization of exponentiated Lomax distribution called the transmuted exponentiated Lomax distribution. The subject distribution is generated by using the quadratic rank transmutation map and taking the exponentiated Lomax distribution as the base distribution. Some mathematical properties along with estimation issues are addressed. The hazard rate function and reliability behavior of the transmuted exponentiated Lomax distribution shows that the subject distribution can be used to model reliability data. We expect that this study will serve as a reference and help to advance future research in the subject area.

REFERENCES

Abdul-Moniem, B.I. and H.F. Abdel-Hameed, 2012. On Exponentiated Lomax Distribution. International Journal of Mathematical Archive, 3(5): 2144-2150.

Aryal, R.G. and C.P. Tsokos, 2009. On The Transmuted Extreme Value Distribution With Application. Nonlinear Analysis: Theory, Methods and Applications, 71: 1401-1407.

Aryal, R.G. and C.P. Tsokos, 2011. Transmuted Weibull Distribution: A Generalization of the Weibull Probability Distribution. European Journal of Pure and Applied Mathematics, 4(2): 89-102.

Aryal, R.G., 2013. Transmuted Log-Logistic Distribution. Journal of Statistics Applications & Probability, 1: 11-20.

Brent, M., 2006. Engineering With Mathcad. Butterworth-Heinemann. Oxford.

Kenney, F.J. and E.S. Keeping, 1962. *Mathematics of Statistics*.Princeton, NJ.

Moors, A.J., 1998. A Quantile Alternative For Kurtosis. Journal of the Royal Statistical Society, D., 37: 25-32.

Shaw, W. and I. Buckley, 2007. The Alchemy of Probability Distributions: Beyond Gram-Charlier Expansions, and a Skew-Kurtotic-Normal Distribution From a Rank Transmutation Map. Research report.