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Estimation Methods for a Two-Parameter Bathtub-Shaped Lifetime Distribution

[1]Josmar Mazucheli, [2]M.E. Ghitany, [3]Francisco Louzada

[1]*Universidade Estadual de Maringa, DEs, PR, Brazil.*
[2]*Department of Statistics & O.R., Faculty of Science, Kuwait University, Kuwait.*
[3]*Universidade de São Paulo, ICMC,SP, Brazil.*

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ABSTRACT

In this paper we compare several methods for estimating the parameters of a two-parameter lifetime distribution that can be used for modeling bathtub hazard rate behavior. The estimation methods considered are: maximum likelihood, ordinary least squares, weighted least squares, Anderson-Darling, and Cramér-von Mises estimation methods. These methods are compared by Monte Carlo simulations in terms of their biases and mean squared errors of the estimated parameters. The simulation study concludes that the Anderson-Darling method is highly competitive with the maximum likelihood method in small and large samples. This conclusion is also supported with the analysis of real data sets.

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INTRODUCTION

In the last years, many probability distributions have been proposed to model bathtub-shaped hazard rates. For reviews about bathtub-shaped hazard rate functions see, for example, Rajarshi and Rajarshi (1988). An interesting two-parameter lifetime distribution used to model bathtub-shaped hazard rate function has been considered by Chen (2000). In general probability distributions used to model bathtub-shaped hazard rate have three or more parameters. As pointed out by Xie *et al*. (2002), models with three or more parameters, considering limited amount of data, may provide inaccurate estimates of its parameters, then distributions with few parameters are important in reliability/survival analysis applications.

Chen (2000) distribution has probability density function (p.d.f.)

$$f(x\,|\,\alpha,\beta)=\alpha\,\beta\,x^{\beta-1}\exp[x^{\beta}+\alpha(1-e^{x^{\beta}})],\qquad x>0,\quad \alpha,\beta>0, \tag{1}$$

where α is a frailty parameter and β is a shape parameter. The p.d.f. (1) is a decreasing (unimodal) function in x for all $\alpha>0$ and $0<\beta<1$ $(\beta\geq 1)$. The modal point can be obtained as a solution in x of the equation $\beta-1+\beta x^{\beta}(1-e^{x^{\beta}})=0.$

The corresponding cumulative distribution function (c.d.f.) and hazard rate function (h.r.f.), respectively, are given by

$$F(x\,|\,\alpha,\beta)=1-\exp[\alpha(1-e^{x^{\beta}})],\qquad x>0,\quad \alpha,\beta>0, \tag{2}$$

$$h(x\,|\,\alpha,\beta)=\alpha\,\beta\,x^{\beta-1}\,e^{x^{\beta}},\qquad x>0,\quad \alpha,\beta>0. \tag{3}$$

The h.r.f. (3) is a bathtub-shaped (increasing) function in x for all $\alpha>0$ and $0<\beta<1$ $(\beta\geq 1)$. The absolute minimum value of $h(x\,|\,\alpha,\beta)$ occurs at the point $x_h=[(1-\beta)/\beta]^{1/\beta}$.

Corresponding Author: M.E. Ghitany, Department of Statistics & O.R., Faculty of Science, Kuwait University, Kuwait.

Xie *et al*. (2002) pointed out that Chen's model has two important features: (1) it has only two parameters to model the bathtub-shaped failure rate function (2) the confidence intervals for the shape parameter and the joint confidence regions for the two parameters have closed form expressions.

In this paper we consider five methods for estimating the parameters of Chen's distribution based on complete sample data. These methods are: maximum likelihood estimation (MLE), the ordinary least-squares estimation (OLSE), weighted least-squares estimation (WLSE), Anderson-Darling estimation (ADE) and Cramér-von Mises estimation (CvME).

The main aim of this paper is to compare the above five estimation methods via intensive simulation studies. Similar studies for other distributions can be found, for example, in Shawky and Bakoban (2012) for the exponentiated gamma distribution, Teimouri *et al*. (2013) for the Weibull distribution, and Usta (2013) for the extended Burr XII distribution.

In Section 2 we discuss the five estimation methods considered in this paper. The comparison of these methods in terms of bias and mean-squared error is presented in Section 3. The five estimation methods are used in fitting two real data sets in Section 4. Some concluding remarks are presented in Section 5.

2. Estimation Methods:

In this section we describe the five considered estimation methods to obtain the estimates of the parameters α and β of Chen distribution.

2.1 Maximum Likelihood Method:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size *n* from Chen distribution with parameters α and β with p.d.f. (1). The maximum likelihood estimates $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$ of the parameters α and β, respectively, are obtained by maximizing, with respect to α and β, the log-likelihood function

$$\ell(\alpha,\beta) = \sum_{i=1}^{n} \ln f(x_i \mid \alpha,\beta) = n \ln(\alpha\beta) + (\beta-1)\sum_{i=1}^{n} \ln(x_i) + \sum_{i=1}^{n} x_i^{\beta} + \alpha \sum_{i=1}^{n} (1 - e^{x_i^{\beta}}). \tag{4}$$

These estimates can also be obtained by solving the score equations:

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} (1 - e^{x_i^{\beta}}) = 0,$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln(x_i) + \sum_{i=1}^{n} x_i^{\beta} \ln(x_i) - \alpha \sum_{i=1}^{n} x_i^{\beta} \ln(x_i)\, e^{x_i^{\beta}} = 0.$$

It follows that

$$\hat{\alpha}_{MLE}(\hat{\beta}_{MLE}) = \frac{n}{\sum_{i=1}^{n} [\exp(x_i^{\hat{\beta}_{MLE}}) - 1]}, \tag{5}$$

where $\hat{\beta}_{MLE}$ is the solution of the non-linear equation

$$\frac{n}{\beta} + \sum_{i=1}^{n} \ln(x_i) + \sum_{i=1}^{n} x_i^{\beta} \ln(x_i) - \hat{\alpha}_{MLE}(\beta) \sum_{i=1}^{n} x_i^{\beta} \ln(x_i)\, e^{x_i^{\beta}} = 0. \tag{6}$$

2.2 Least squares methods:

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the order statistics of a random sample of size *n* from a distribution with c.d.f. $F(x)$. It is well known that

$$E[F(X_{i:n})] = \frac{i}{n+1}, \qquad Var[F(X_{i:n})] = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

2.2.1 Ordinary least squares method:

For the Chen distribution, the least square estimates $\hat{\alpha}_{OLSE}$ and $\hat{\beta}_{OLSE}$ of the parameters α and β, respectively, are obtained by minimizing the function:

$$S(\alpha,\beta) = \sum_{i=1}^{n} \left[F(x_{i:n}) - \frac{i}{n+1} \right]^2. \tag{7}$$

These estimates can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}\left[F(x_{i:n}\,|\,\alpha,\beta)-\frac{i}{n+1}\right]\Delta_1(x_{i:n}\,|\,\alpha,\beta)=0, \tag{8}$$

$$\sum_{i=1}^{n}\left[F(x_{i:n}\,|\,\alpha,\beta)-\frac{i}{n+1}\right]\Delta_2(x_{i:n}\,|\,\alpha,\beta)=0. \tag{9}$$

where

$$\Delta_1(x\,|\,\alpha,\beta)=-(1-e^{x^{\beta}})\exp[\alpha(1-e^{x^{\beta}})],\quad \Delta_2(x\,|\,\alpha,\beta)=\alpha\, x^{\beta}\ln(x)\ \exp[x^{\beta}+\alpha(1-e^{x^{\beta}})]. \tag{10}$$

2.2.2 Weighted least squares method:

The weighted least-squares estimates $\hat{\alpha}_{WLSE}$ and $\hat{\beta}_{WLSE}$ of the parameters α and β, respectively, are obtained by minimizing the function:

$$W(\alpha,\beta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n})-\frac{i}{n+1}\right]^2. \tag{11}$$

These estimates can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}\frac{1}{i(n-i+1)}\left[F(x_{i:n}\,|\,\alpha,\beta)-\frac{i}{n+1}\right]\Delta_1(x_{i:n}\,|\,\alpha,\beta)=0, \tag{12}$$

$$\sum_{i=1}^{n}\frac{1}{i(n-i+1)}\left[F(x_{i:n}\,|\,\alpha,\beta)-\frac{i}{n+1}\right]\Delta_2(x_{i:n}\,|\,\alpha,\beta)=0, \tag{13}$$

where $\Delta_1(.\,|\,\alpha,\beta)$ and $\Delta_2(.\,|\,\alpha,\beta)$ are given by (10) .

2.3 Minimum distance methods:

In this subsection we present two estimation methods for of the parameters α and β, based on minimization of goodness-of-fit statistics. This class of statistics is based on the difference between the estimate of the cumulative distribution function and the empirical distribution function.

2.3.1 Anderson-Darling method:

The Anderson-Darling estimates $\hat{\alpha}_{ADE}$ and $\hat{\beta}_{ADE}$ of the parameters α and β, respectively, are obtained by minimizing, with respect to α and β, the function:

$$A(\alpha,\beta)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\,\ln\left\{F(x_{i:n}\,|\,\alpha,\beta)\left[1-F(x_{n+1-i:n}\,|\,\alpha,\beta)\right]\right\}. \tag{14}$$

These estimates can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(x_{i:n}\,|\,\alpha,\beta)}{F(x_{i:n}\,|\,\alpha,\beta)}-\frac{\Delta_1(x_{n+1-i:n}\,|\,\alpha,\beta)}{1-F(x_{n+1-i:n}\,|\,\alpha,\beta)}\right]=0, \tag{15}$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(x_{i:n}\,|\,\alpha,\beta)}{F(x_{i:n}\,|\,\alpha,\beta)}-\frac{\Delta_2(x_{n+1-i:n}\,|\,\alpha,\beta)}{1-F(x_{n+1-i:n}\,|\,\alpha,\beta)}\right]=0, \tag{16}$$

where $\Delta_1(.\,|\,\alpha,\beta)$ and $\Delta_2(.\,|\,\alpha,\beta)$ are given by (10) .

2.3.2 Cramér-von Mises method:

The Cramér-von Mises estimates $\hat{\alpha}_{CvME}$ and $\hat{\beta}_{CvME}$ of the parameters α and β, respectively, are obtained by minimizing, with respect to α and β, the function

$$C(\alpha,\beta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:n}\,|\,\alpha,\beta)-\frac{2i-1}{2n}\right]^2. \tag{17}$$

These estimates can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}\left[F(x_{i:n}\,|\,\alpha,\beta)-\frac{2i-1}{2n}\right]\Delta_1(x_{i:n}\,|\,\alpha,\beta)=0, \tag{18}$$

$$\sum_{i=1}^{n}\left[F(x_{i:n}\,|\,\alpha,\beta)-\frac{2i-1}{2n}\right]\Delta_2(x_{i:n}\,|\,\alpha,\beta)=0, \tag{19}$$

where $\Delta_1(.\,|\,\alpha,\beta)$ and $\Delta_2(.\,|\,\alpha,\beta)$ are given by (10) .

3. Simulations:

In this section we present results of some numerical experiments to compare the performance of the five estimators discussed in the previous section. We have taken sample sizes n=20, 30, … , 120, and parameter values (α,β) : (0.5, 0.5), (0.5, 5), (5, 0.5), (5,5). Figures 1-2, respectively, show the shapes of the p.d.f. and h.r.f. corresponding to the parameter values used in the simulations.

For each combination (n,α,β), we have generated N=100,000 pseudo-random samples from the two-parameter bathtub distribution using the inverse c.d.f. method:

$$x=F^{-1}(u\,|\,n,\alpha,\beta)=\left[\ln\left(1-\frac{\ln(1-u)}{\alpha}\right)\right]^{1/\beta},$$

where u is a uniform (0,1) observation.

The estimates were obtained in Ox version 6:20, (see Doornik, 2007), using MaxBFGS function. To assess the performance of the methods, we calculated the bias and the mean-squared error for the simulated estimates of the parameters α and β:

$$Bias(\hat{\theta})=\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i-\theta), \qquad MSE(\hat{\theta})=\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i-\theta)^2, \qquad \theta=\alpha,\beta .$$

Figures 3-4 show, respectively, the bias of the simulated estimates of α and β. From these two figures, we observe that:

(1) all estimators of α are *positively biased*,
(2) all estimators of β are *positively biased* except the OLSE of β is *negatively biased*,
(3) the MLE or OLSE of α has smaller bias compared to other estimators,
(4) the OLSE of β has the smallest absolute bias compared to other estimators,
(5) the biases of all estimators of α and β tend to zero for large n, i.e. the estimators are asymptotically unbiased for the parameters.

Figures 5-6 show, respectively, the MSE of the simulated estimates of α and β. From these two figures, we observe that

(1) the MSE of the ADE of α is the smallest among all other estimators,
(2) the MSE of the ADE of β is the smallest among all other estimators when $n \le 40,$ otherwise the MSE of the MLE of β is the smallest among other estimators,
(3) the MSE of all estimators of α and β tend to zero for large n, i.e. all estimators are consistent for the parameters.

4. Applications:

In this section we analyze two real data sets for comparing the considered five estimation methods for the Chen distribution.

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Data set 1: (Fiegl and Zelen 1965)

This data set represents the survival times (in weeks) of 17 AG-positive patients (identified by the presence of Auer rods and/or significant granulator of the leukemia cells in the bone marrow at diagnosis) who died of acute myelogenous leukemia:
65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65.

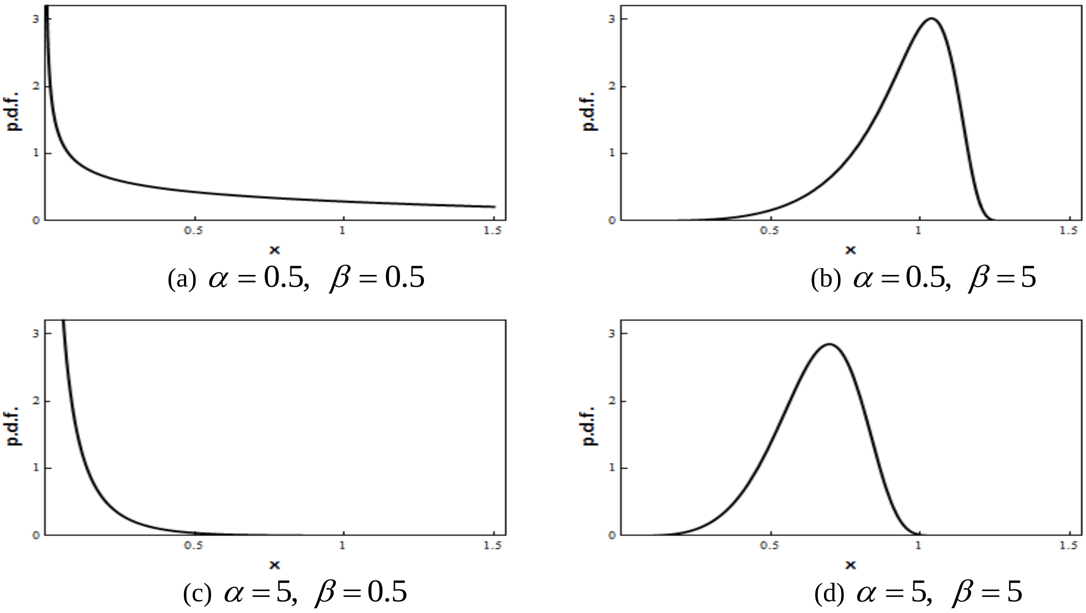

Fig. 1: Shapes of the probability density function used in the simulations.

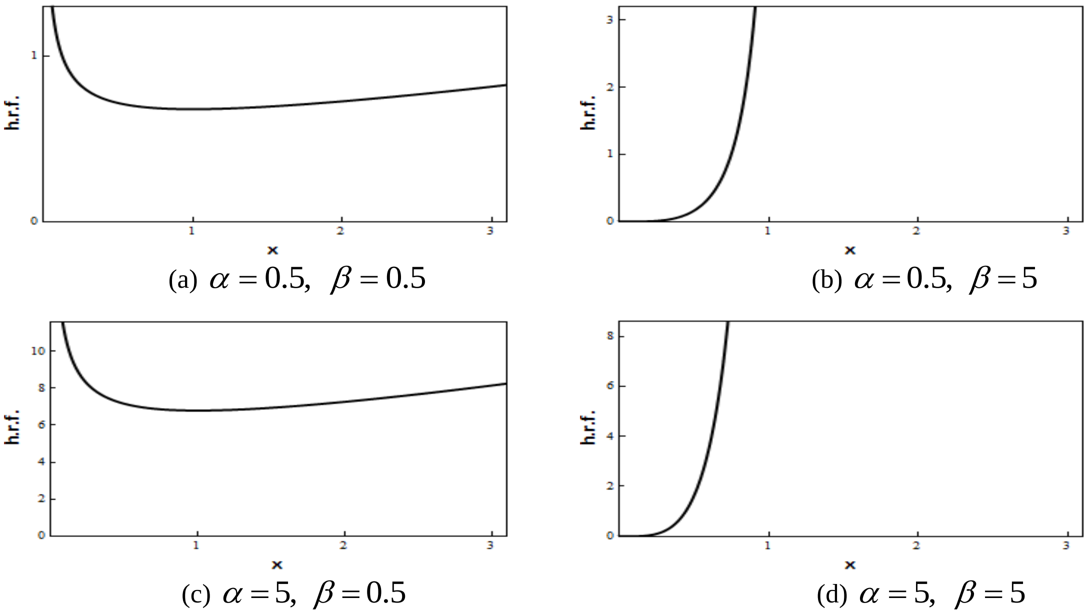

Fig. 2: Shapes of the hazard rate function used in the simulations.

Data set 2: (Lawless 2003)

This data set represents the number of 1000s of cycles to failure for 60 electrical appliances in a life test:
0.014, 0.034, 0.059, 0.061, 0.069, 0.080, 0.123, 0.142, 0.165, 0.210, 0.381, 0.464, 0.479, 0.556, 0.574, 0.839, 0.917, 0.969, 0.991, 1.064, 1.088, 1.091, 1.174, 1.270, 1.275, 1.355, 1.397, 1.477, 1.578, 1.649, 1.702, 1.893, 1.932, 2.001, 2.161, 2.292, 2.326, 2.337, 2.628, 2.785, 2.811, 2.886, 2.993, 3.122, 3.248, 3.715, 3.790, 3.857, 3.912, 4.100, 4.106, 4.116, 4.315, 4.510, 4.580, 5.267, 5.299, 5.583, 6.065, 9.701.
Figure 7 shows the empirical scaled TTT-transform (Aarset 1987):

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$$T(r/n) = \frac{\sum_{i=1}^{r} x_{i:n} + (n-r)x_{r:n}}{\sum_{i=1}^{n} x_{i:n}}, \qquad r = 1,2,\ldots,n, \tag{20}$$

for data sets 1 and 2. The figure also shows that each transform changes behavior from convex to concave, indicating a bathtub shaped failure rate function of the underlying population from which the data is drawn.

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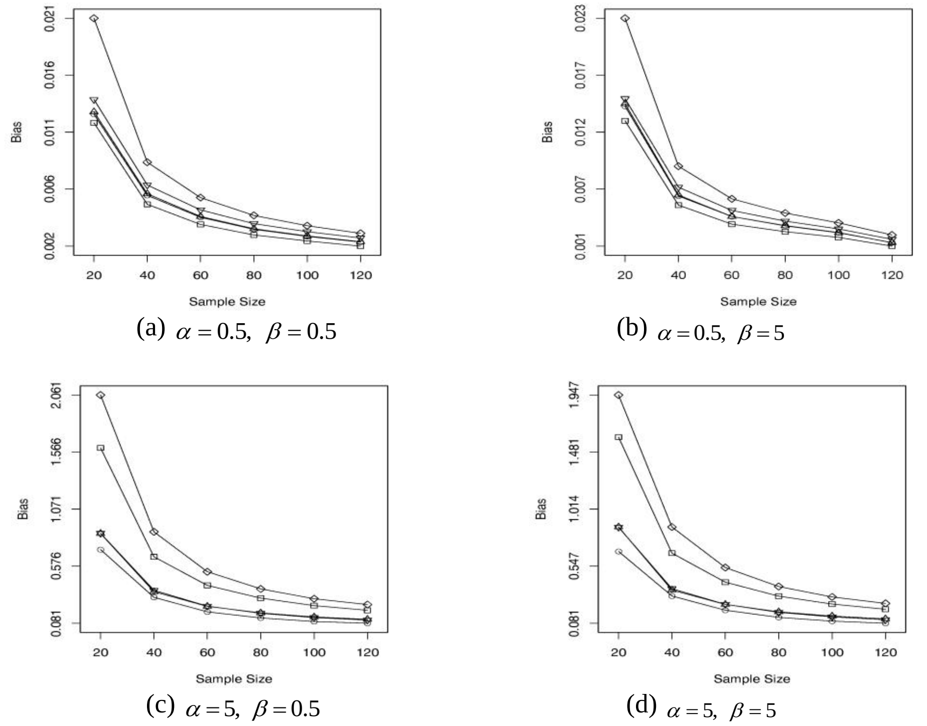
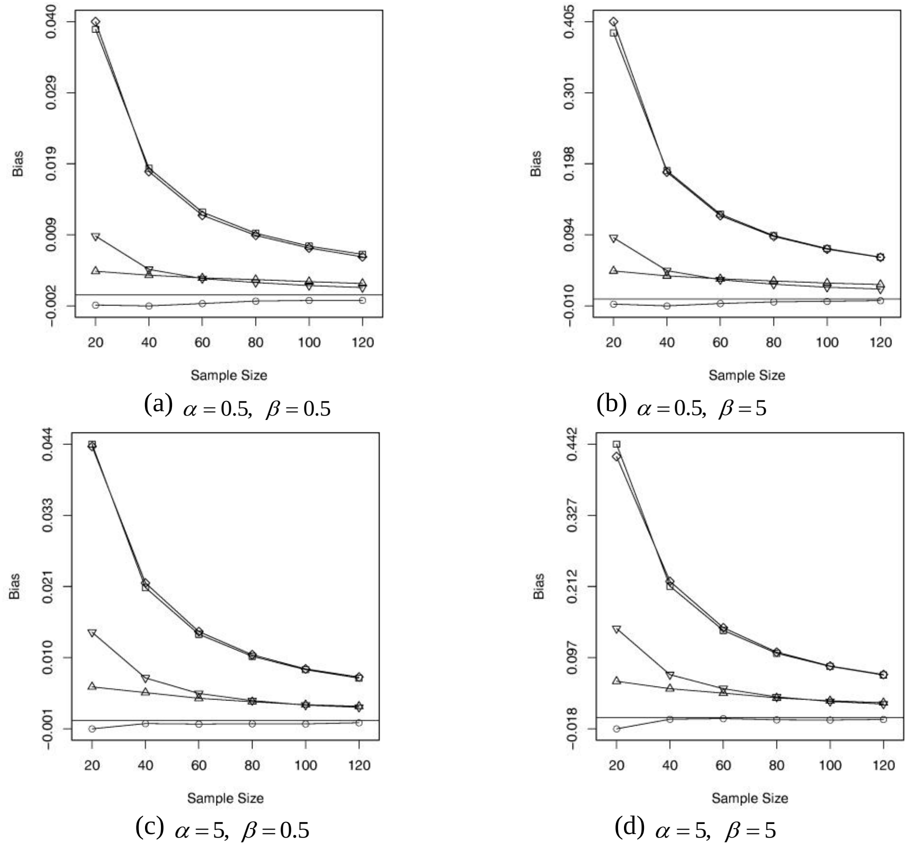


(a) $\alpha = 0.5,\ \beta = 0.5$ (b) $\alpha = 0.5,\ \beta = 5$

(c) $\alpha = 5,\ \beta = 0.5$ (d) $\alpha = 5,\ \beta = 5$

Fig. 3: Bias of $\hat{\alpha}$ (□ : MLE, ○ : OLSE, Δ: WLSE, ◊: CvME, ∇ : ADE).



(a) $\alpha = 0.5,\ \beta = 0.5$ (b) $\alpha = 0.5,\ \beta = 5$

(c) $\alpha = 5,\ \beta = 0.5$ (d) $\alpha = 5,\ \beta = 5$

Fig. 4: Bias of $\hat{\beta}$ (□ : MLE, ○ : OLSE, Δ: WLSE, ◊: CvME, ∇ : ADE).

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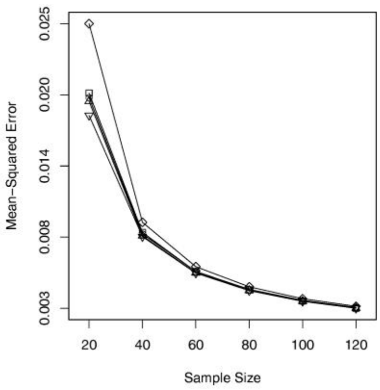
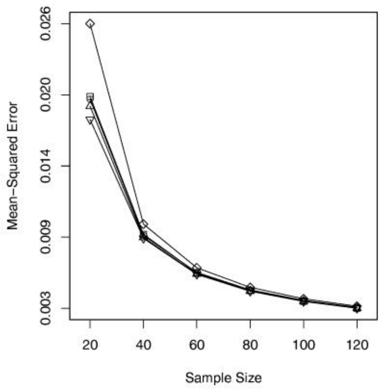
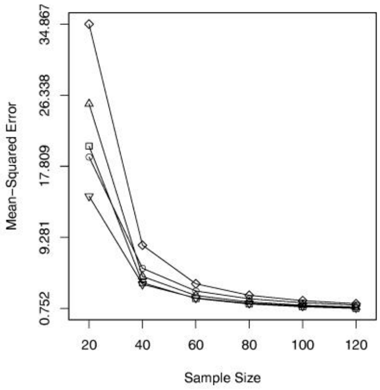


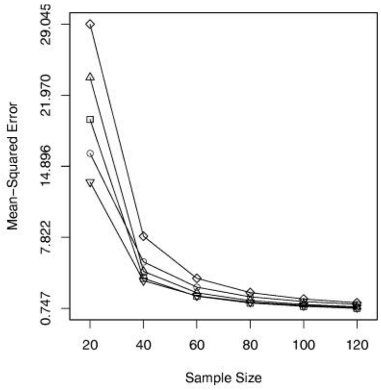
(a) $\alpha = 0.5,\ \beta = 0.5$



(b) $\alpha = 0.5,\ \beta = 5$

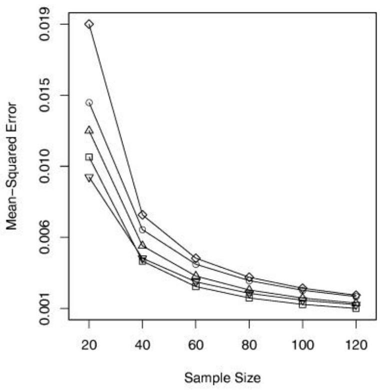


(c) $\alpha = 5,\ \beta = 0.5$

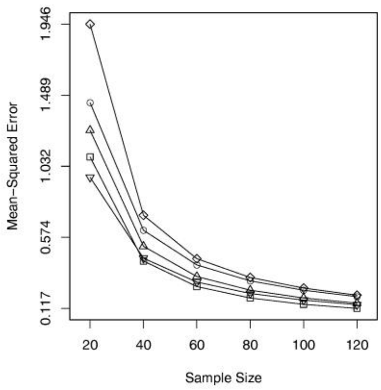


(d) $\alpha = 5,\ \beta = 5$

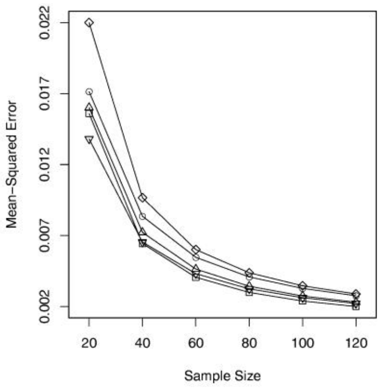
Fig. 5: MSE of $\hat{\alpha}$ (□ : MLE, ○ : OLSE, Δ: WLSE, ◊: CvME, ∇ : ADE).



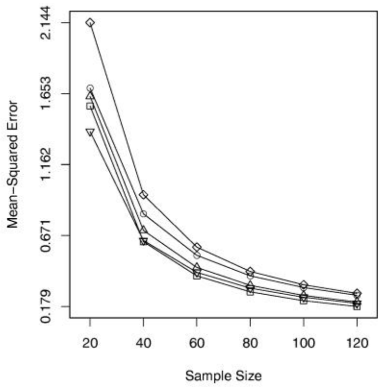
(a) $\alpha = 0.5,\ \beta = 0.5$



(b) $\alpha = 0.5,\ \beta = 5$

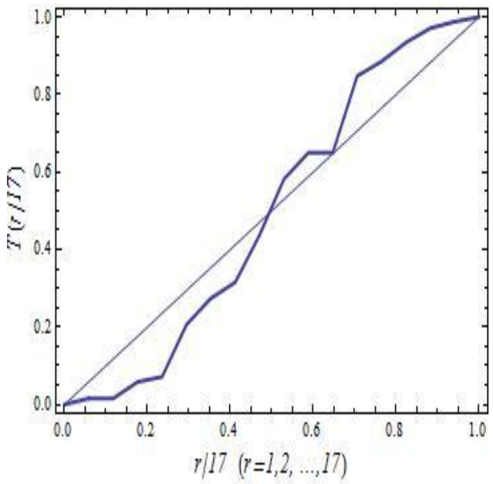


(c) $\alpha = 5,\ \beta = 0.5$

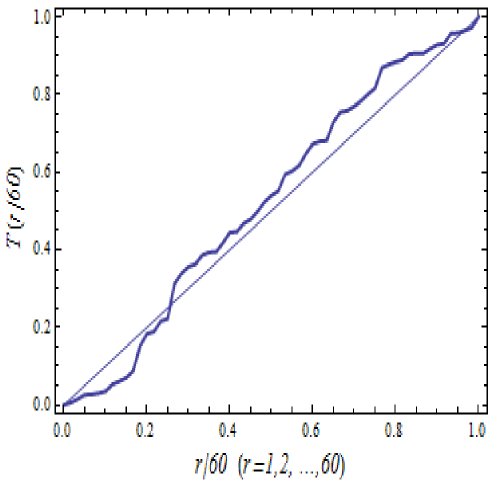


(d) $\alpha = 5,\ \beta = 5$

Fig. 6: MSE of $\hat{\beta}$ (□ : MLE, ○ : OLSE, Δ: WLSE, ◊: CvME, ∇ : ADE).



(a) Data set 1



(b) Data set 2

Fig. 7: Empirical scaled total time on test for data sets 1 and 2.

Table 1 (2) shows that the Anderson-Darling (Cramér-von Mises) estimation method has the smallest statistic and highest p-value of the K-S goodness-of-fit test. That is, the Anderson-Darling (Cramér-von Mises) estimation method is the preferred method for data set 1 (2) among the five considered estimation methods.

Table 1: Parameters estimates and K-S goodness-of-fit test for data set 1.

	Estimated Parameters		K-S test	
Method	$\widehat{\alpha}$	$\widehat{\beta}$	statistic	p-value
MLE	0.0295	0.2991	0.1358	0.8720
OLSE	0.0481	0.2614	0.1376	0.8619
WLSE	0.0439	0.2710	0.1122	0.9664
ADE	0.0402	0.2768	0.1062	0.9798
CvME	0.0414	0.2731	0.1160	0.9560

Table 2: Parameters estimates and K-S goodness-of-fit test for data set 2.

	Estimated Parameters		K-S test	
Method	$\widehat{\alpha}$	$\widehat{\beta}$	statistic	p-value
MLE	0.2452	0.5318	0.0577	0.9817
OLSE	0.2413	0.5346	0.0533	0.9922
WLSE	0.2410	0.5396	0.0528	0.9930
ADE	0.2404	0.5426	0.0521	0.9941
CvME	0.2379	0.5453	0.0492	0.9972

5. Conclusions:

In this paper we compared, via intensive simulation experiments, the estimation of the parameters of the two-parameter bathtub distribution using five estimation methods, namely the maximum likelihood, ordinary least-squares, weighted least-squares, Anderson-Darling and Cramér-von Mises. The simulation study concludes that the Anderson-Darling estimation method is highly competitive with the maximum likelihood estimation method for most practical sample sizes. This conclusion is also supported with the analysis of real data sets.

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