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Modified Block Method for the Direct Solution of initial Value Problems of Fourth order Ordinary Differential Equations.

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ABSTRACT

In this article, we present a new block method for the direct solution of initial value problems of fourth order ordinary differential equations. The approach of collocation approximation is adopted in the derivation of the main scheme with continuous coefficients, from where additional schemes were developed. The implementation strategy is by combining the main scheme and the additional schemes as simultaneous integrator to initial value problem of fourth order ordinary differential equations. Properties analysis of the block showed that it is consistent, convergent, zero stable and absolutely stable. Our method was tested with numerical examples solved using existing method and was found to give better results.

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INTRODUCTION

The initial value problems of fourth
Order ordinary differential equation:

$$y^{(iv)} = f\left(x, y, y', y'', y'''\right), y(x_0) = \eta_0, \\ y'(x_0) = \eta_1, y''(x_0) = \eta_2, y'''(x_0) = \eta_3 \quad (1)$$

is considered. It is assumed that the numerical solution to (1) is required on a given set of mesh: $\prod = \{x_n / x_n = a + nh, h = x_{n+1} - x_n, n = 0,1,2,...N\}$ Where $N = \frac{b-a}{h}$.

Many researchers has solved (1), they include: Oman (1999), Kayode (2008), Olabode (2009) and Adesanya (2011) who developed block methods for numerical solution of fourth order ordinary differential equations. The works of Olabode (2009) and Adesanya (2011) serve as improvement on the work of Kayode (2008) who developed Linear multistep method for the solution of fourth order ordinatry differential equations whose implementation is Predictor – Corrector mode.

Block method have been shown to eliminate the drawbacks of Predictor corrector method as discussed in Olabode (2009) and Yusuph (2004), consequently, we are motivated to advance the course of research work in this area.

Anake (2011), Bolaji (2012), Bolaji et al (2012 a and b) in their works have proposed single Step hybrid methods for the direct numerical solution of initial value problems of second order and third order Ordinary differential equations respectively. In all Cases, their methods of implementation is block mode with the proposed methods being efficient, adequate and suitable towards catering for the class of problem- higher order ordinary differential equations - for which they were designed. The success story of the afore – mentioned works is that a single step method was shown and effectively tested to be capable of solving higher order ordinary differential equations, against the initial believe, before now, that single step numerical methods were only capable of solving first order ordinary differential equations.

Thus, in this work, we are motivated to go further in the proposition of a single step method for the direct numerical solution of higher order ordinary differential equations, which eliminates the use of predictors by providing sufficiently accurate simultaneous difference equations from a single continuous formula

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and its derivatives.

According to Awoyemi et al (2011), the general block formula is given by:

$$Y_m = ey_n + h^{\mu} df(y_n) + h^{\mu} bF(y_m) \tag{2}$$

where e is $s \times s$ vector, d is r - vector and b is $r \times r$ vector, s is the interpolation points and r is the *collection points. F is a k – vector whose* J^{th} entry is $f_{n+j} = f(t_{n+j}, y_{n+j})$, μ is the order of the differential equation.
Given a predictor equation in the form:

$$Y_m^{(0)} = ey_n + h^{\mu} df(y_n). \tag{3}$$

By Putting (3) in (2) we have:

$$Y_m = ey_n + h^{\mu} df(y_n) + h^{\mu} bF(ey_n + h^{\mu} dfy_n). \tag{4}$$

Equation (4) is called a self starting block-predictor-corrector method because the prediction equation is gotten directly from the block formula (Shampine & Watts (1969) and (Kayode (2008)).

Consequently, our focus in this paper is the proposition of an implicit continuous hybrid block – Predictor corrector algorithm with a single step length for the solution of initial value problems of fourth order ordinary differential equations.

2. Methodology:

2.1 Derivation of the continuous coefficients:

We take our basis function to be a power series of the form:

$$y(x) = \sum_{j=0}^{r+s} a_j x^j \tag{5}$$

The third derivative of (5) gives:

$$y^{(iv)}(x) = \sum_{j=0}^{r+s} j(j-1)(j-2)(j-3)a_j x^{j-4} \tag{6}$$

By putting (6) into (1) we have the differential system:

$$\sum_{j=0}^{r+s} j(j-1)(j-2)(j-3)a_j x^{j-4} \tag{7}$$

$$= f(x, y(x), y'(x), y''(x), y'''(x))$$

Where a_j are the parameters to be determined, while r+s denotes the number of collocation and interpolation points. By collocating (7) at the mesh points $x = x_{n+j}, j = 0(\frac{1}{4})1$, and interpolating (5) at $x = x_{n+j}$, $j = 0, \frac{1}{4}(\frac{1}{4})\frac{3}{4}$ yields a system of equations:

$$\sum_{j=0}^{r+s} a_j x^j = y_{n+s} \tag{8}$$

$$\sum_{j=0}^{r+s} j(j-1)(j-2)(j-3)a_j x^{j-4} = f_{n+r} \tag{9}$$

By putting these system of equations in matrix form and then solved to obtain the values of parameters a_j 's , j = 0, $\frac{1}{4}$, .. which when substituted in (5), yields, after some manipulation, an hybrid linear method with continuous coefficients of the form:

$$y(x) = \sum_{j=0}^{1} \alpha_j y_{n+j}(x) + h^4 \sum_{j=0}^{1} \beta_j f_{n+j}(x) \tag{10}$$

The co efficient of $\alpha_j(x)$ and β_j are:

$$\alpha_0(t) = \frac{1}{6}(t^3 + 3t^2 + 2t)$$

$$\alpha_{\frac{1}{4}}(t) = -\frac{1}{3}(t^3 + 9t^2 + 2t)$$

$$\alpha_{\frac{1}{2}}(t) = -\frac{1}{2}(t^3 + 10t^2 + t)$$

$$\alpha_{3/4}(t) = -\tfrac{1}{8}\left(22t^3 + 6t^2 + t + 3\right)$$
$$\alpha_1(t) = \tfrac{1}{2}\left(2t^3 + t^2 + t\right)$$
$$\beta_0(t) = \frac{1}{184{,}320}\left(7t^8 + t^7 - 4t^6 + 2t^5 - 3t^4 + t^3 + 4t - 7\right)$$
$$\beta_{1/4}(t) = \frac{1}{184{,}320}\begin{bmatrix} 5t^8 - 3t^7 - 124t^6 + 9t^5 \\ -5t^4 - 9t^3 + 3t \end{bmatrix}$$
$$\beta_{1/2}(t) = \frac{1}{184{,}320}\begin{bmatrix} 17t^8 - 36t^7 - 72t^6 - 300t^5 - \\ 174t^4 + 72t^3 - 17t + 36 \end{bmatrix}$$
$$\beta_{3/4}(t) = \frac{1}{184{,}320}\begin{bmatrix} 84t^8 - 6t^7 + 64t^6 - 100t^5 - \\ 24t^4 - 64t^3 + 6t - 84 \end{bmatrix}$$
$$\beta_1(t) = \frac{1}{184{,}320}\begin{bmatrix} 4t^8 + 9t^7 - 12t^6 + 5t^5 - \\ 4t^4 - 4t^3 + 12t - 9 \end{bmatrix} \qquad (10)$$

Where $t = \dfrac{x - x_n}{h}$

2.2 Derivation of the Block Method:

The general block formula proposed by Awoyemi et al (2011), in the Normalized form is given by:

$$A^{(0)}Y_m = ey_n + h^{\mu-\lambda}df\left(y_n\right) + h^{\mu-\lambda}bF\left(y_m\right) \qquad (11)$$

By evaluating (10) at $t = 1$; the first, second and the third derivative at $x = x_{n+1}, i = 0(\tfrac{1}{4})1$ and substituting into (10) gives the coefficients of (11) as:

$$d = \begin{bmatrix} \begin{bmatrix} \frac{3373}{30965760} & \frac{9472}{7741440} & \frac{15957}{3440640} & \frac{4505}{3870720} & \frac{2034}{1290240} & \frac{10592}{1290240} & \frac{25758}{1290240} & \frac{47616}{1290240} \end{bmatrix} \\ \frac{31195}{1958400} \; \frac{11872}{322560} \; \frac{49}{2560} \; \frac{9}{90} \; \frac{257}{2880} \; \frac{551}{6840} \; \frac{27}{320} \; \frac{9}{90} \end{bmatrix}^T$$

$$e = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tfrac{1}{4} & \tfrac{1}{2} & \tfrac{3}{4} & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tfrac{1}{32} & \tfrac{1}{8} & \tfrac{9}{32} & \tfrac{1}{2} & \tfrac{1}{4} & \tfrac{1}{2} & \tfrac{3}{4} & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \tfrac{1}{384} & \tfrac{1}{48} & \tfrac{9}{128} & \tfrac{1}{6} & \tfrac{1}{32} & \tfrac{1}{8} & \tfrac{1}{32} & \tfrac{1}{2} & \tfrac{1}{4} & \tfrac{1}{2} & \tfrac{3}{4} & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

$A^0 = 16\times16$ identity matrix

$$B = \begin{bmatrix} \frac{2780}{30965760} & \frac{15104}{7741440} & \frac{34668}{3440640} & \frac{114688}{3870720} & \frac{2140}{1290240} & \frac{21248}{1290240} & \frac{67068}{1290240} & \frac{139264}{1290240} \\ \frac{-8472}{30965760} & \frac{-10107}{7741440} & \frac{-11492}{3440640} & \frac{-15675}{3870720} & \frac{-2365}{1290240} & \frac{-8809}{1290240} & \frac{-9877}{1290240} & \frac{5015}{1290240} \\ \frac{716}{30965760} & \frac{2816}{7741440} & \frac{4860}{3440640} & \frac{16384}{3870720} & \frac{516}{1290240} & \frac{3328}{1290240} & \frac{8100}{1290240} & \frac{24576}{1290240} \\ \frac{-131}{30965760} & \frac{-512}{7741440} & \frac{-891}{3440640} & \frac{-2560}{3870720} & \frac{-94}{1290240} & \frac{-608}{1290240} & \frac{1458}{1290240} & \frac{-2560}{1290240} \end{bmatrix}$$

$$\begin{bmatrix} \frac{45900}{1958400} & \frac{32256}{322560} & \frac{156}{2560} & \frac{24}{90} & \frac{-264}{2880} & \frac{2356}{6840} & \frac{102}{320} & \frac{32}{90} \\ \frac{-23970}{1958400} & \frac{-6735}{322560} & \frac{18}{2560} & \frac{6}{90} & \frac{-264}{2880} & \frac{456}{6840} & \frac{72}{320} & \frac{12}{90} \\ \frac{9860}{1958400} & \frac{3584}{322560} & \frac{20}{2560} & \frac{8}{90} & \frac{106}{2880} & \frac{76}{6840} & \frac{42}{320} & \frac{32}{90} \\ \frac{-1792}{1958400} & \frac{-672}{322560} & \frac{-3}{2560} & 0 & \frac{-19}{2880} & \frac{-18}{6840} & \frac{-3}{320} & \frac{7}{90} \end{bmatrix}^T$$

3. Analysis of the Properties of the block:

3.1 Order of the Method:

The linear operator of the block (11) is defined as:

$$L\{y(x):h\} = Y_m - ey_m + h^{\mu-\lambda}df\left(y_m\right) + h^{\mu-\lambda}bF\left(y_m\right) \qquad (12)$$

By expanding $y(x_n + ih)$ and $f(x_n + jh)$ in Taylor series, (12) becomes:

$$L\{y(x):h\} = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \dots \\ + C_p h^p y^{(p)}(x) + \tag{13}$$

The block (11) and associated linear operator are said to have order p if $C_0 = C_1 = \dots = C_{p+1} = 0, C_{p+2} \neq 0.$

The term C_{p+2} is called the error constant and implies that the local truncation error is given by:

$$t_{n+k} = C_{p+2} h^{(p+2)} y^{(p+2)}(x_n) + 0h^{(p+3)} \tag{14}$$

Hence the block (11) has order 8 with error constant:

$$C_{p+2} = \begin{bmatrix} \frac{1355}{6193152}, \frac{1129}{2580480}, \frac{1129}{1720320}, \frac{71131}{81283860}, \\ \frac{6117}{671420}, \frac{2417}{321560}, \frac{5227}{1638200}, \frac{6226}{31180194}, \\ \frac{2104}{891019}, \frac{1835}{287923}, \frac{1292}{298142}, \frac{2155}{893287}, \\ \frac{5105}{1935152}, \frac{4221}{2819456}, \frac{1942}{1834231}, \frac{28182}{341940} \end{bmatrix}^T$$

3.2 Zero Stability of the Block:

The block (11) is said to be Zero stable if the roots $z_s = 1,2,\dots, N$ of the characteristic polynomial $\rho(z) = \det(zA - E)$, satisfies $|z| \leq 1$ and the root $|z| = 1$ has multiplicity not exceeding the order of the differential equation. Moreover as $h^{\mu} \to 0, \rho(z) = z^{r-\mu}(\lambda - 1)$,

Where μ is the order of the differential equation, for the block (11), $r = 16, \mu = 4$

$$\rho(z) = \lambda^{12}(\lambda - 1)^4 = 0$$

$$\Rightarrow \lambda = 0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1$$

Hence our method is Zero stable.

3.3 Convergence:

The necessary and sufficient condition for a numerical method to be convergent is for it to be Zero stable and has order $p \geq 1$, from the above condition, it could be seen that our method is convergent.

4. Numerical Experiments:

To test the accuracy, workability and suitability of the method, we adopted our method to solving some initial value problems of fourth order ordinary differential equations.

Test Problem 1:

We consider a non linear fourth order problem:

$$y^{(iv)} = (y')^2 - y(y'') - 4x^2 + e(1 - 4x + x^2)$$

$$0 \leq x \leq 1, y(0) = 0, y'(0) = 1, y''(0) = 3, y'''(0) = 1,$$

$$h = \frac{1}{32}$$

Whose exact solution is given by:

$$y(x) = \sin 10x + \cos 10x.$$

The result is as shown in table 1.

Test Problem 2:

We consider an homogenous initial value problem of fourth order ordinary differential equation:

$$y^{(iv)} + y'' = 0$$

$$y(0) = y'(0) = \frac{1.1}{72 - 50\pi}, y''(0) = \frac{1}{144 - 100\pi},$$

$$y'''(0) = \frac{1.2}{144 - 100\pi}; \quad h = \frac{0.1}{32}$$

Whose exact solution is:

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$$y(x) = \frac{1 - x - x\cos x - 1.2\sin x}{144 - 100\pi}$$

Our method was used to solve the problem and result compared with Kayode (2008). The result is as shown in table I.

4.2 Numerical Results:

We make use of the following Notations in the table of results:
XVAL: Value of the independent variable where numerical value is taken.
ERC: Exact result at XVAL
NRC: Our Numerical result at XVAL
ERR: Error of our result at XVAL.

5. Discussion of Results:

In this paper, we have proposed an Implicit Hybrid Block – Predictor Corrector algorithm for the numerical solution of initial value problems of fourth order ordinary differential equations. For better performance of the method, step size is chosen within the stability interval. The results of our new method when compared with the block method proposed by Kayode (2008) showed that our method is more accurate.

Table 1: Showing results for problem 1.

XVAL	ERC	NRC	ERR
0.103125	1.119264744787591900	1.119347875160145673	7.3198321E-13
0.206250	1.271599493198048300	1.2216387790134546012	1.2171610E-11
0.306250	1.452110907065012200	1.4522066080124645600	6.0874881E-10
0.406250	1.666216862500120800	1.667222187345305300	1.937419E -10
0.506250	1.915347109920913400	1.915256609021376700	4.804318E -10
0.603125	2.201081767908965600	2.201181770124546500	1.016991E -10
0.703125	2.514440293337009000	2.514430967800341900	1.894379E - 09
0.803125	2.877516387746618300	2.877490600912456550	3.849295E - 09
0.903125	3.282936158805117400	3.282926122460345600	5.497805E - 09
1.003125	3.733049511495201100	3.732982001568356200	8.652849E - 09

Table 2: Showing results for problem 2.

XVAL	ERC	NRC	ERR	ERR in Kayode(2008)
0.103125	0.001300799589367158	0.001300798534101687	3.713118E-18	0.49873299E- 15
0.206250	0.002531773700195635	0.002531754312431673	6.301613E-17	0.67654215E -15
0.306250	0.003652478978884993	0.003652481412435763	3.105345E -16	0.31350790E- 14
0.406250	0.0046959532231804849	0.004696041232415678	9.317469E -16	0.94360283E- 13
0.506250	0.0056576423608034461	0.005657569022134261	2.174123E -15	0.22116856E- 13
0.603125	0.006507754608034524	0.006507708202140167	4.327921E -15	0.43379362E -13
0.703125	0.007298314767638522	0.007298279021345601	7.766013E -15	0.77870869E- 13
0.803125	0.007998520222728983	0.007998460132450184	1.281345E -14	0.12863494E -12
0.903125	0.008607246703302495	0.008607181023675676	1.982614E -14	0.19927115E- 12
1.003125	0.009124283967030094	0.0091242876102456617	2.936478E -14	0.29323245E -12

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