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N-Dimensional Stokes-Brinkman Equations using a Mixed Finite Element Method

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ABSTRACT

A macroscale model is developed to model a porous medium and adjacent free fluid. Typically, fluid flows through a porous medium by a pressure gradient. In this problem, we introduce the model that fluid moves by the movement of solid phases. Hybrid mixture theory (HMT) and nondimensionalization are used to obtain the macroscale generalized Darcy's law called the Brinkman equation. This generalization helps to match the tangential stress acting on the fluid at the free-fluid/porous-medium interface. We apply the Stokes equation for incompressible slow flow in the adjacent free fluid domain. A variational formulation of the Stokes-Brinkman equations is provided by using a mixed finite element method for n-dimensional domain. Applications of this model include modeling fluid flow through moving solid phases such as animal hair and hairlike structures.

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INTRODUCTION

Nowadays, there are several problems involving the fluid flow through a porous medium and adjacent free-fluid region. Classical porous-media flow problems involve a static phase with a liquid-phase pressure gradient inducing fluid flow. In this research, we develop governing equations that will be used to model a fluid flowing due to the movement of the solid phase such as biological hairlike structures. The configuration is illustrated in Figure 1.

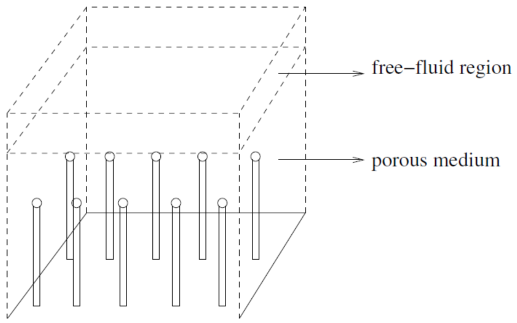

Fig. 1: A snapshot of a cell of the porous medium and adjacent free-fluid region when the angle between cylinders and horizontal plane is 90 degrees.

Typically, the Stokes or Navier-Stokes equations are used in free-fluid region and Darcy's law with the Beavers-Joseph condition in the porous medium (Hanspal, N.S., *et al.*, 2006; Arbogast, T., D.S. Brunson., 2007; Layton, W.J., *et al*., 2003). However, because of the limitation of the boundary condition when Darcy's law is applied, in this study, we use the Stokes-Brinkman equations which are employed by several authors, cf. e.g., (Neale, G., W. Nader., 1974; Martys, N.S., J.G. Hagedorn., 2002; Martys., N.S., 2001; Kirsh., V.A., 2006).

We next clarify the difference between Darcy's law and Brinkman equations. Darcy's law (Darcy, H., 1856).

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$$v_D = -\frac{k}{\mu}[\nabla p - \rho g], \tag{1}$$

where v_D is Darcy's velocity; k is the permeability tensor; μ is the dynamic viscosity; p is the pressure; ρ is the fluid density in the porous medium and g is the gravity, is typically employed where viscosity and inertial effects are negligible while Brinkman (1947a) claimed that in some cases the viscous shearing stresses acting on the fluid are not negligible and additional term should be included. To rigorously determine the macroscale form of Brinkman equations, we use hybrid mixture theory (HMT) (Cushman, J.H., *et al*., 2002; Bennethum, L.S., J.H. Cushman., 1996; Weinstein., T.F., 2005) and nondimensionalization to obtain form of the Brinkman equations, see Section 2:

$$\mu k^{-1} \cdot \left(\varepsilon^l v^l - \varepsilon^l v^s\right) + \nabla p - \frac{\mu}{\varepsilon^l} \nabla \cdot \left(2\varepsilon^l d^l\right) = \rho g, \tag{2}$$

which is the equation used in the porous medium where k^{-1} is the inverse of the permeability tensor; ε^l is the porosity; v^l and v^s are the velocities of the liquid and solid phases, respectively; $d^l = 0.5\left(\nabla v^l + \left(\nabla v^l\right)^T\right)$ is the rate of deformation tensor and the superscript T is the transpose. For the divergent-free continuity equation, equation (2) is consistent with [11, equation (16)] except that (2) includes the term $\varepsilon^l v^s$ on the left-hand side whereas in (Weinstein., T.F., 2005) it is assumed that the velocity of the solid phase is zero. The extra term $\left(\mu / \varepsilon^l\right)\nabla \cdot \left(2\varepsilon^l d^l\right)$ comes from the liquid phase stress tensor and with this term the generalized Darcy's law is called the Brinkman equations. This term helps to match the tangential stress acting on the fluid at the free-fluid/porous-medium interface.

Note that for incompressible fluid, if we let the inverse of the permeability k^{-1} go to zero in the free-fluid region and ε^l be a constant in space, the equation (2) becomes

$$\nabla p - \rho g = \frac{\mu}{\varepsilon^l} \Delta\left(\varepsilon^l v^l\right), \tag{3}$$

which is the Stokes equation typically derived from Navier-Stokes equation with the porosity $\varepsilon^l = 1.$

We now have a system of equations in free-fluid region when the Stokes equation is coupled with the divergent-free continuity equation while in porous medium domain we have the Brinkman equation (2) coupling with the continuity equation (Weinstein, T.F., L.S. Bennethum., 2006):

$$\varepsilon^l + \left(1 - \varepsilon^l\right)\nabla \cdot \left(\varepsilon^l \left(v^l - v^s\right)\right) = 0, \tag{4}$$

where ε^l is the material time derivative of the porosity with respect to the solid phase, $\varepsilon^l = \partial\varepsilon^l / \partial t + v^s \cdot \nabla\varepsilon^l$. In Section 2, the macroscopic scale of Brinkman equation which are specific forms of the momentum equation are derived. The model discretization of the Stokes-Brinkman system of the Stokes-Brinkman equations using a mixed finite element method is shown in Section 3. Finally, the conclusion is drawn in Section 4.

Macroscale Brinkman Equation:

We derive the macroscale Brinkman equation by using hybrid mixture theory (HMT) and nondimensionalization method. HMT starts with microscale equations and uses averaging theorem to derive macroscopic field equations and then exploits the entropy inequality to derive constitutive equations. Transferring a microscale variable to a macroscale variable is defined in term of the intrinsic phase average. That is the average of the microscale variable weighted by the volume of the phase. The macroscale conservation of momentum balance for liquid phase, l, using HMT is provides as follows. For more details, see (Chamsri., K., 2012). The conservation of momentum is

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$$\rho\left(\frac{\partial v^l}{\partial t}+v^l\cdot\nabla v^l\right)+\mu k^{-1}\cdot\left(\varepsilon^l v^l-\varepsilon^l v^s\right)+\nabla p-\frac{\mu}{\varepsilon^l}\nabla\cdot\left(2\varepsilon^l d^l\right)=\rho g. \tag{5}$$

Next, we normalize equation (5) and neglect some terms in the equation with respect to a slow motion of the fluid. To normalize (5), we choose scaling parameters: characteristic length L, characteristic frequency f, characteristic speed v_0, reference pressure p_0 and gravitational acceleration g_0. By defining the following dimensionless variables

$$\tilde{v}^l=\frac{v^l}{v_0},\tilde{v}^s=\frac{v^s}{v_0},\tilde{p}=\frac{p}{p_0},\tilde{g}=\frac{g}{g_0}, \tag{6}$$

$$\tilde{\nabla}=L\nabla,\tilde{t}=ft,\tilde{\Delta}=L^2\Delta. \tag{7}$$

The dimensionless form of equation (5) is

$$\varepsilon^l\tilde{v}^l-\varepsilon^l\tilde{v}^s=-\frac{k}{\mu}\left[\rho f\frac{\partial\tilde{v}^l}{\partial\tilde{t}}+\frac{\rho v_0}{L}\tilde{v}^l.\tilde{\nabla}\tilde{v}^l+\frac{p_0}{Lv_0}\tilde{\nabla}\tilde{p}-\frac{\rho g_0}{v_0}\tilde{g}-\frac{1}{\varepsilon^l}\frac{\mu}{L^2}\tilde{\nabla}\cdot\left(2\varepsilon^l\tilde{d}^l\right)\right]. \tag{8}$$

For a slow flow problem, for example biological hairlike structures such as animal hair, we choose the reference time t to be the time it takes for a periodic array of cylinders to go through one cycle, L to be the height of the array of cylinders, ρ and μ to be the density and dynamic viscosity of water at temperature $40^\circ C$ and g_0 to be the Earth's gravity. Then the time-dependent and nonlinear terms are relatively small in comparison with other terms. Neglecting these terms, (5) becomes

$$\mu k^{-1}\cdot\left(\varepsilon^l v^l-\varepsilon^l v^s\right)+\nabla p-\frac{\mu}{\varepsilon^l}\nabla\cdot\left(2\varepsilon^l d^l\right)=\rho g, \tag{9}$$

which is the Brinkman equation.

Model Discretization:

Discretization of the model is the first step leading to numerical solutions. In this section, we apply a mixed finite element method to discretize the continuity equation (4) and the Stokes-Brinkman equations (9). Before discretizing the equations, we first note that

$$\nabla\cdot\left(2\varepsilon^l d^l\right)=\Delta v+\nabla\left(\nabla\cdot v\right), \tag{10}$$

where $\varepsilon^l v^l=v$ and ε^l is assumed to be a constant, i.e. the porosity does not change in space. Define the spaces

$$L_0^2\left(\Omega\right)=\left\{q\in L^2\left(\Omega\right):\int_\Omega q\,d\Omega=0\right\}, \tag{11}$$

where $L_0^2\left(\Omega\right)$ is the Sobolev space. The weak formulation of the system of equations is as follows: Find $\left(v,p\right)\in H^1\left(\Omega\right)\times L_0^2\left(\Omega\right)$ such that

$$\mu k^{-1}\cdot v+\nabla p-\frac{\mu}{\varepsilon^l}\Delta v=\rho g+\mu k^{-1}\cdot\varepsilon^l v^s+\frac{\mu}{\varepsilon^l}\Delta f \tag{12}$$

$$\nabla\cdot v=f, \tag{13}$$

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where $f = -\varepsilon^l / \left(1-\varepsilon^l\right) + \nabla \cdot \varepsilon^l v^s; H^1(\Omega)$ is the Hilbert space; and Ω is a computational domain. Writing (12) as scalar equations in n-dimensions gives: for $i = 1, 2, ..., n,$

$$\mu\left[k_{ij}^{-1}v_j\right] - \frac{\mu}{\varepsilon^l}\left[\frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j}\right)\right] + \frac{\partial p}{\partial x_i} = \rho g_i + \mu\varepsilon^l\left[k_{ij}^{-1}v_j^s\right] + \frac{\mu}{\varepsilon^l}\frac{\partial f}{\partial x_i}, \tag{14}$$

for $j = 1, 2, ..., n,$ where gravity is given by $g = (0, 0, -g)$ and the repeated index j within a single term indicates summation. To find the weak formulation, we multiply (14) by test functions $w_i \in H_0^1(\Omega), i = 1, 2, ..., n$ and integrate over the domain Ω. This yields

$$\int_\Omega \left(\mu\left[k_{ij}^{-1}v_j\right] - \frac{\mu}{\varepsilon^l}\left[\frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j}\right)\right] + \frac{\partial p}{\partial x_i}\right) w_i d\Omega = \int_\Omega \left(\rho g_i + \mu\varepsilon^l\left[k_{ij}^{-1}v_j^s\right] + \frac{\mu}{\varepsilon^l}\frac{\partial f}{\partial x_i}\right) w_i d\Omega. \tag{15}$$

Note that the repeated index i in (15) indicates the number of equations, not the summation. Integrating by parts the second-order term, the pressure term and the source term f, we have the weak formulation: Find $(v, p) \in H^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega \mu\left[k_{ij}^{-1}v_j\right] w_i d\Omega + \frac{\mu}{\varepsilon^l}\int_\Omega \left[\frac{\partial v_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}\right] d\Omega - \int_\Omega p\frac{\partial w_i}{\partial x_i} d\Omega \\
&= \int_\Omega \left(\rho g_i + \mu\varepsilon^l\left[k_{ij}^{-1}v_j^s\right]\right) w_i d\Omega - \frac{\mu}{\varepsilon^l}\int_\Omega f\frac{\partial w_i}{\partial x_i} d\Omega - \int_\Gamma p w_i n_i d\Gamma \\
&+ \frac{\mu}{\varepsilon^l}\int_\Gamma \left[\frac{\partial v_i}{\partial x_j} n_j\right] w_i d\Gamma + \frac{\mu}{\varepsilon^l}\int_\Gamma f w_i n_i d\Gamma, \qquad \forall w_i \in H_0^1(\Omega), i = 1, 2, ..., n,
\end{aligned} \tag{16}$$

where Γ is the boundary of the domain Ω and $n_i, i = 1, 2, ..., n,$ is the outward unit normal vector. Let T_h be a triangulation of domain Ω and

$$V_h = \left\{v \in H^1(\Omega) : v|_K \textit{ is quadratic}, \forall K \in T_h\right\} \tag{17}$$

$$H_h = \left\{q \in L_0^2(\Omega) : q|_K \textit{ is linear}, \forall K \in T_h\right\} \tag{18}$$

be finite-dimensional subspaces of $H^1(\Omega)$ and $L_0^2(\Omega)$, respectively. The approximate solutions $(v_i, p) \in V_h \times H_h$ in finite element method are as follows.

Let

$$v_i(x) = \sum_{m=1}^{M} \psi_m(x) v_i^m = \Psi^T V_i, \tag{19}$$

$$p(x) = \sum_{l=1}^{L} \phi_l(x) p_l = \Phi^T P_i, \tag{20}$$

where V_i and P are vectors of the velocities and pressure, respectively; ψ_m and ϕ_l are called basis functions; Ψ and Φ are their vector forms and the integers M and L are determined by the interpolation

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function. For example, for a tetrahedral element, $M=10$ for quadratic for the velocity v_i and $L=4$ for linear function for the pressure p. Substituting the basis function Ψ into w_i and (19) and (20) into (16), we have

$$\left(\mu\int_\Omega k_{ij}^{-1}\Psi\Psi^T d\Omega\right)V_j+\frac{\mu}{\varepsilon^l}\int_\Omega\left(\frac{\partial\Psi}{\partial x_j}\frac{\partial\Psi^T}{\partial x_j}d\Omega\right)V_i\left(-\int_\Omega\frac{\partial w_i}{\partial x_i}d\Omega\right)P$$

$$=\int_\Omega\left(\rho g_i+\mu\varepsilon^l k_{ij}^{-1}v_j^s\right)\Psi d\Omega-\frac{\mu}{\varepsilon^l}\int_\Omega f\frac{\partial\Psi}{\partial x_i}d\Omega\left(-\int_\Gamma\Psi\Phi^T n_i d\Gamma\right)P$$

$$+\frac{\mu}{\varepsilon^l}\left(\int_\Gamma\Psi\frac{\partial\Psi^T}{\partial x_j}n_j d\Gamma\right)V_i+\frac{\mu}{\varepsilon^l}\int_\Gamma f\Psi n_i d\Gamma. \tag{21}$$

Let Ω_1 and Ω_2 be a porous medium domain and the domain of the adjacent free-fluid region, respectively. Therefore $\Omega=\Omega_1\cup\Omega_2$. We first form an element matrix in domain Ω_2 . Define

$$\tilde{A}=\int_{\Omega_2^e}\Psi\Psi^T d\Omega_2^e,\ \tilde{K}_{ij}=\int_{\Omega_2^e}\left(\frac{\partial\Psi}{\partial x_j}\frac{\partial\Psi^T}{\partial x_j}d\Omega_2^e\right),\tilde{Q}_i=\int_{\Omega_2^e}\Phi\frac{\partial\Psi^T}{\partial x_j}d\Omega_2^e, \tag{22}$$

$$\tilde{F}_i=\int_{\Omega_2^e}\left(-\rho g_i+\mu\varepsilon^l k_{ij}^{-1}v_j^s\right)\Psi d\Omega_2^e-\frac{\mu}{\varepsilon^l}\int_{\Omega_2^e}f\frac{\partial\Psi}{\partial x_j}d\Omega_2^e-\left(\int_{\Gamma_2^e}\Psi\Phi^T n_i d\Gamma_2^e\right)P$$

$$+\frac{\mu}{\varepsilon^l}\left(\int_{\Gamma_2^e}\Psi\frac{\partial\Psi^T}{\partial x_j}n_i d\Gamma_2^e\right)V_i+\frac{\mu}{\varepsilon^l}\int_{\Gamma_2^e}f\Psi n_i d\Gamma_2^e, \tag{23}$$

where Ω_2^e is the element domain such that $\Omega_2=\bigcup_e\Omega_2^e$. Then (21) becomes

$$\mu k_{ij}^{-1}\tilde{A}V_j+\left(\mu/\varepsilon^l\right)\left(\tilde{K}_{ij}\right)V_i-\tilde{Q}_i^T P=\tilde{F}_i, \tag{24}$$

where again the superscript T denotes the transpose. Applying the same process to the continuity equation (13) which is:

$$\frac{\partial v_j}{\partial x_j}=-\frac{\varepsilon^l}{1-\varepsilon}+\varepsilon^l\frac{\partial v_j^s}{\partial x_j}, \tag{25}$$

where have the weak form

$$-\int_{\Omega_2^e}\Phi\frac{\partial\Psi^T}{\partial x_j}d\Omega_2^e V_j=-\int_{\Omega_2^e}\left(\frac{\varepsilon^l}{1-\varepsilon^l}+\varepsilon^l\frac{\partial v_j^s}{\partial x_j}\right)\Phi d\Omega_2^e. \tag{26}$$

Let

$$\tilde{F}_4=-\int_{\Omega_2^e}\left(-\frac{\varepsilon^l}{1-\varepsilon^l}+\varepsilon^l\frac{\partial v_j^s}{\partial x_j}\right)\Phi d\Omega_2^e. \tag{27}$$

Therefore (26) becomes

$$-\tilde{Q}_j V_j=\tilde{F}_4. \tag{28}$$

Define $\tilde{B} = \left(\mu / \varepsilon^l\right)\left(\tilde{K}_{ij}\right)$ and recall that the repeated index j is the summation over $j,\ j = 1, 2, ..., n.$ The element matrix form of the system of equations (24) and (28) is as follows,

$$\begin{pmatrix} \mu k_{11}^{-1}\tilde{A} + \tilde{B} & \mu k_{12}^{-1}\tilde{A} & \mu k_{13}^{-1}\tilde{A} & -\tilde{Q}_1^T \\ \mu k_{21}^{-1}\tilde{A} & \mu k_{22}^{-1}\tilde{A} + \tilde{B} & \mu k_{23}^{-1}\tilde{A} & -\tilde{Q}_2^T \\ \mu k_{31}^{-1}\tilde{A} & \mu k_{32}^{-1}\tilde{A} & \mu k_{33}^{-1}\tilde{A} + \tilde{B} & -\tilde{Q}_3^T \\ -\tilde{Q}_1 & -\tilde{Q}_2 & -\tilde{Q}_3 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ P \end{pmatrix} = \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \\ \tilde{F}_4 \end{pmatrix} \tag{29}$$

We now have the matrix form of the discrete system of equations in domain Ω_2 .

Next, we find the element matrix form in the free-fluid region, domain Ω_1 . Because of no solid phases in this region, we have no permeability and the porosity becomes one in this domain. Therefore, the momentum equations are the same as those in Ω_2 except there are no source term, permeability and the porosity is one. Applying the same process as that applied to obtain (29), we have

$$\frac{\mu}{\varepsilon^l}\left(\int_{\Omega_1^e} \frac{\partial \Psi}{\partial x_j}\frac{\partial \Psi^T}{\partial x_j} d\Omega_1^e\right)V_i - \left(\int_{\Omega_1^e} \frac{\partial \Psi}{\partial x_i}\Phi^T d\Omega_1^e\right)P = \left(-\int_{\Gamma_1^e} \Psi\Phi^T n_i d\Gamma_1^e\right)P + \frac{\mu}{\varepsilon^l}\left(\int_{\Gamma_2^e} \Psi \frac{\partial \Psi^T}{\partial x_j} n_i d\Gamma_1^e\right)V_i, \tag{30}$$

where Ω_1^e be the element domain such that $\Omega_1 = \bigcup_e \Omega_1^e$. Writing (30) and (28) into a matrix form, we have

$$\begin{pmatrix} \tilde{B} & 0 & 0 & -\tilde{Q}_1^T \\ 0 & \tilde{B} & 0 & -\tilde{Q}_2^T \\ 0 & 0 & \tilde{B} & -\tilde{Q}_3^T \\ -\tilde{Q}_1 & -\tilde{Q}_2 & -\tilde{Q}_3 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ P \end{pmatrix} = \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \\ \tilde{F}_4 \end{pmatrix}, \tag{31}$$

where

$$\tilde{B}_i = \left(-\int_{\Gamma_2^e} \Psi\Phi^T n_i d\Gamma_1^e\right)P + \frac{\mu}{\varepsilon^l}\left(\int_{\Gamma_2^e} \Psi \frac{\partial \Psi^T}{\partial x_j} n_i d\Gamma_1^e\right)V_i. \tag{32}$$

Note that the velocities and pressure in Ω_1 and Ω_2 are different except at the free-fluid/porous-medium interface but we still use the same notations for simplicity.

Conclusion:

We develop macroscale Stokes-Brinkman equations for coupled free-fluid/porous-medium viscous flow using Hybrid Mixture Theory and nondimensionalization. Typically, pressure gradient induces fluid flowing through a porous medium. It should be noted that to the author’s knowledge this is the first time the porous medium equations are being used to model a fluid flowing due to the movement of the solid phase. We use a mixed finite element method to discretize the Stokes-Brinkman system of equations for n-dimensional domain by using the indicial notation introduced in the previous sections. For more details about indicial notations, see (Bennethum., L.S., 2011). Numerical solutions of this model using a mixed finite element method will be provided in future work.

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