Australian Journal of Basic and Applied Sciences, 10(8) April2016, Pages: 133-142

AUSTRALIAN JOURNAL OF BASIC AND APPLIED SCIENCES

ISSN:1991-8178 EISSN: 2309-8414
Journal home page: www.ajbasweb.com

Copositive Approximation by Using Whitney Theorem in $L_{p,\mu}(I)$,$1 \le p < \infty$

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ARTICLE INFO

Article history:
Received 12 February 2016
Accepted 18 March 2016
Available online 20 April 2016

Keywords:

ABSTRACT

In this paper, the relationship between the degree of best approximation and an averaged module of smoothness of order $k$in the space $L_{p,\mu}[a,b]$,$1 \le p < \infty$,where $I \subseteq R$ and $I = [a,b]$have been discussed. The estimation between the degree of best approximation by algebraic polynomial of degree$\le k-1$, and modulus of smoothness of degree$\le k$ to the function $f \in X \cap Y$, where $X = L_{p,\mu}(I)$ and $Y = \Delta^0(J_r)$ and $1 \le p < \infty$ have been found

INTRODUCTION

Hasslerwhitney is one of the scientist who accomplished Whitney theorem in approximation theory, which provides the following : (for $f \in L_p[a,b], 0 < p \le \infty$, then there exists $q_{k-1} \in \Pi_{k-1}$,a polynomial of degree $\le k-1$,such that

$$\|f - q_{k-1}\|_{L_p[a,b]} \le c\omega_k(f, b-a, [a,b])_p$$

Many reasercher proved this theorem, Burkill(1952) when $(k=2, p=\infty)$, Whitney(Burkill(1952),Whitney.(1959)) when ($p=\infty$), Brudnyi(1964) when ($1 \le p < \infty$),Storozhenko(1978) when ($0 < p < 1$), and K.A.kopotun proved the Whitney theorem of type k-monotone functions .

In(2003) E.S. Bhaya(2003) proved in theorem (A) the Whitney theory of interpolators type for k-monotone functions for K.A.kopotun

Theorem A:

for $m, k \in N, m < k$ and $f \in \Delta^k \cap W_p^m(I)$.Then for any, $n \ge k-1$, there exists a polynomial $p_n \in \Pi_n$ such that :

$$\left\|f^{(j)} - p_n^{(j)}\right\|_p \le c(p,k)\omega_{k-j}^{\varphi}(f^{(j)}, n^{-1}, I)_p \quad \text{for } j = 1,\dots,m .$$

The classical Whitney theorem establishes the equivalence between the modulus of smoothness $\omega_r(f, |I|, I)_p$ and the error of best approximation $E_s(f)_p$of a function $f : I \to R$ by algebraic polynomials of degree$\le s-1$ in $L_p, 1 \le p < \infty$

ToCite ThisArticle: Saheb K.AL-Saidy and Nagham Ali Hussen.,Copositive Approximation by Using Whitney Theorem in 〖L〗_(p,μ) (I) ,1≤p<∞. *Aust. J. Basic & Appl. Sci., 10(8): 133-142, 2016*

Australian Journal of Basic and Applied Sciences, 10(8) April2016, Pages: 133-142

Proposition(B) (Kassim, N.M.,):

Let Q be an arbitrary polynomials and let $P_{m-1}(Q)$ be a polynomial of degree$\leq m-1$ interpolating Q at m points inside $[a,b]$, then for every $0<p\leq\infty$

$\|P_{m-1}(Q)\|_{p[a,b]}\leq c\|Q(.)\|_{p[a,b]}$

Lemma (C) (Anton , H.,2002):

Let f be a continuous function on $I=[a,b]$, then

$$\int_a^b f(x)dx=\int_{a+C_1}^{b+C_1} f(x-C_1)\,dx or \int_a^b f(x)dx=\int_{a-C_1}^{b-C_1} f(x+C_1)\,dx$$

C_1 is absolute

2-μ -*Best Approximation:*

The space $L_{p,\mu}(I)$ of all μ −measurable function , where $1\leq p<\infty$, $I\subseteq R$, $I=[a,b]$, b is a positive integer and $\mu: I\longrightarrow R^{+}\cup\{a,b\}$ is a measurable function can be defined as

$$L_{p,\mu}(I)=\left\{f,f:[a,b]\subseteq R\longrightarrow R:\left(\int_I |f(t)|^p d\mu(x)\right)^{\frac{1}{p}}<\infty\right\}$$

$$where\ 1\leq p<\infty$$

And the (quasi) norm $\|f\|_{L_{p,\mu(I)}}<\infty$

$$\|f\|_{L_{p,\mu(I)}}=\left(\int_I |f(t)|^p d\mu(t)\right)^{\frac{1}{p}},\ t\in I \qquad \textbf{(2.1)}$$

For a function f in the space $L_{p,\mu}(I)$,$1\leq p<\infty$ and the measurable positive function μ we have $f(t)d\mu(t)\geq 0$ for every $f(t)\geq 0$ and $t\in I$

The polynomials used in our work differ in the form and according to the degree of what we want to achieve in the proof . for $r\geq 0$ and let $J_r=\{j_i\}_{i=1}^{r}$ be the collection of point , so that a$=J_0<J_1<\cdots<j_r<j_{r+1}=b$, we set $P_n(t)=\prod_{i=1}^{r}(t-j_1)$. and we let $Y=\Delta^0(J_r)$ be the set of functions f which change their sign exactly at the point $j_i\in J_r$and we will write $f\in\Delta^0$. This assumption is equivalent to $f(t)\prod(t,J_r)\geq 0$
$a\leq t\leq b$.

3- *Modulus of Smoothness:*

The moduli of smoothness are intended for mathematicians working in approximation theory, numerical analysis and real analysis. Measuring the smoothness of a function by differentiability is too crude for many purposes in approximation theory. More subtle measurement are provided by modulus of smoothness. We will use modulus of smoothness which are connected with difference of higher orders.

The k-th symmetric difference of the function *f*(Sendov, B. and opove, V. A., 1983)is defined by

$\Delta_h^k(f,t,I)_\mu=$

$$\Delta_h^k(f,t)_\mu=\begin{cases}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}f\left(t-\frac{kh}{2}+ih\right), & t\mp\frac{kh}{2}\in I\\ 0\ , otherwise\end{cases}$$

Where

$\binom{k}{i}=\frac{k!}{i!(k-i)!}$, is the binomial coefficient

The k-th local module of smoothness (Abdul Naby, Z. E., 2008) of a function$f\in L_{p,\mu\,(I)}$, is defined by

$$\omega_k(f,\delta,I)_{p,\mu}=\underbrace{sup}_{0<h\leq\delta}\left\|\Delta_h^k(f,.)\right\|_{L_{p,\mu(I)}}\ ,\delta>0 \qquad \textbf{(3.1)}$$

$$\omega_k(f,\delta,I)_{p,\mu}=\underbrace{sup}_{0<h\leq\delta}\left\|\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}f\left(t-\frac{kh}{2}+ih\right)\right\|_{L_{p,\mu(I)}}$$

The so called τ − modulus (or sendov-popov modulus) (Ditzian, Z. and V. Totik, 1987), an averaged modulus of smoothness, defined for bounded measurable function f on an interval I by.

$\tau_k(f,\delta,I)_{p,\mu}=\|\omega_k(f,.,\delta)\|_{L_{p,\mu(I)}}$ Where

$\omega_k(f,t,\delta)_\mu=sup\left\{\left|\Delta_h^k(f,y)\right|: y\mp\frac{kh}{2}\in\left[t-\frac{kh}{2},t+\frac{kh}{2}\right]\cap I\right\}$ is the k-th local modulus of smoothness of *f*(Sendov, Bl. and opove, V. A., 1988). From the definition one can easily see

$\tau_k(f,\delta,I)_{\mu,\infty} = \omega$

A new way of measuring smoothness was introduced by Ditzian–Totik (Ditzian, Z. and V. Totik, 1987).The Ditzian - Totik modulus of smoothness of $f \in L_{p,\mu(I)}$, $1 \le p < \infty$ which is define for such an f as follows.

$$\omega_k^{\varphi}(f,\delta,I)_{p,\mu} = \underbrace{sup}_{|h|<\delta} \left\|\Delta_{h\varphi(.)}^k(f,.)\right\|_{L_{p,\mu(I)}} , I = [a,b] \quad \textbf{(3.2)}$$

4- Chebyshev Partition:

We have used in this paper the following notations and facts also the partition of period ℓ_i , therefore we found Chebyshev partition, which takes the form:

$X_j = a\cos\frac{j\pi}{n}$, $a =$ positive integer such that $1 \le a < \infty$, $0 \le j \le n$,

to an interval I . Now we denote

$I_j = [X_{j+1}, X_j]$

h_j is a length of I_i that is $h_j = |I_j| = X_j - X_{j+1}$,$0 \le j \le n$

and $\Delta_n(t) = \frac{\varphi(t)}{n} + \frac{1}{n^2}$, and $c_1\Delta_n(t) \le h_j \le c_2\Delta_n(t)$ for $t \in I_j$ and c_1, c_2 are constant.

Let $J_r = \{j_1, \ldots, j_r : j_0 = a < j_1 < \cdots < j_r < b = j_{r+1}\}$ we denote by $\Delta^0(J_r)$ the set of all functions $f \in X \cap Y$ has $0 \le r < \infty$ change sign $k-$ times in J_r(Leviatan, D., 1996),in particular if $r = 0$, then $\Delta^0 = \Delta^0(J_0)$ denotes the set of all nonnegative functions on $[a,b]$.

For

$\delta = min|j_{i+1} - j_i|$ where$j_0 = a and j_{r+1} = b$.

If $j_i \in (X_{j(i)+1}, X_{j(i)}) i = 1, \ldots, r$ then it is convenient to denote $j_i^{(z)} \le X_{j(i)+1}$ and $j_i^{(k-1)} \ge X_{j(i)}$,$k > 1$ such that

$j_i' < j_i'' < \cdots < j_i^{(k-1)}$ that is $j_i \in (j_i^{(z)}, j_i^{(k-1)}) z = 1,2, \ldots, k-2$

$\ell_i = [j_i^{(z)}, j_i^{(k-1)}]$ and $J_i = \left[\frac{j_i + j_i^{(z)}}{k-1}, \frac{j_i + j_i^{(k-1)}}{k-1}\right]$

then

$c_1 h_j \le |\ell_i| = (k-1)|J_i| \le c_2 h_j$, where $c_i, i = 1,2, \ldots$,r positive number.

and therefore , we get the following facts which we used to prove many results

$$|\ell_i| \approx |J_i| \approx h_j \approx \Delta_n(t) \text{ also } n|\ell_i| \approx n\Delta_n(t) \approx \varphi(t) \text{ for } t \in \ell_i \quad \textbf{(4.1)}$$

We would like to point out that the symbol ℓ_i ,$z = 1, \ldots, k-1$ not represent a derivatives but a symbols of a set of points that exist between $X_{j(i)}$ and $X_{j(i)+1}$,meaning within an period ℓ_i , and $I = \bigcup_{i=1}^{r} \ell_i$, we proved many results andtheories on the period ℓ_i, and the fact that the periods ℓ_i ,$i = 1, \ldots, r$ isomorphic and have the same properties .So is the proof of these results is true on the aggregate period I .

In (Bhaya, E.S., 2003), recall that for any continuous function f on $[a,b]$ there exist an algebraic polynomial p_{k-1} of degree $\le k-1$ interpolating f inside $[a,b]$,such that

$$\|f - p_{k-1}\|_{L_p[a,b]} \le c(p)\omega_k(f, b-a, [a,b])_p \quad \textbf{(4.2)}$$

5-*Auxiliary Results:*

We need the following lemmas to prove the main result

Lemma (5.1):

Let $f \in L_{P,\mu}(I)$be unbounded function, $1 \le P < \infty$, let$J = [a_1, b_1] \subset I$

Define $I_1 = [a_1, b_1 + h]$, where $h \in \left[0, \frac{b_1 - a_1}{k}\right)$ let $k = 1,2, \ldots$.If $I_1 \subset I$ then

$$\|f(.)\|_{P,I_1,\mu} \le C\omega_k(f, h, I_1)_{P,\mu} + C\|f(.)\|_{P,J,\mu} \quad \textbf{(5.1.1)}$$

Proof:

Since $\Delta_h^k f(x) = \sum_{0 \le i \le k} \binom{k}{i} (-1)^{k-i} f(x + ih)$

then

$$\Delta_h^k f(x) = f(x+kh) + \sum_{0\le i\le k-1} (-1)^{k-i}\binom{k}{i} f(x+ih)$$

$$f(x+kh) = \Delta_h^k f(x) - \sum_{0\le i\le k-1} (-1)^{k-i}\binom{k}{i} f(x+ih)$$

If we take the absolute value for two sides of equation, we get

$$|f(x+kh)| = \left|\Delta_h^k f(x) - \sum_{0\le i\le k-1} (-1)^{k-i}\binom{k}{i} f(x+ih)\right|$$

$$\le \left|\Delta_h^k f(x)\right| + \left|\sum_{0\le i\le k-1} \binom{k}{i} f(x+ih)\right|$$

Integrate of both sides of equation, we get

$$\left(\int_{b_1-kh}^{b_1-(k-1)h} |f(x+kh)|^P\, d\mu(x)\right)^{\frac{1}{P}} \le \left(\int_{b_1-kh}^{b_1-(k-1)h} \left|\Delta_h^k f(x)\right|^P d\mu(x)\right)^{\frac{1}{P}} + \left(\int_{b_1-kh}^{b_1-(k-1)h} \sum_{0\le i\le k-1} |f(x+ih)|^P\, d\mu(x)\right)^{\frac{1}{P}}$$

Since $[b_1-kh, b_1-(k-1)h] \subset I_1$, then

$$\le \left(\int_{I_1} \left|\Delta_h^k f(x)\right|^P d\mu(x)\right)^{\frac{1}{P}} + \left(\sum_{0\le i\le k-1} \int_{b_1-kh}^{b_1-(k-1)h} |f(x+ih)|^P\, d\mu(x)\right)^{\frac{1}{P}}$$

$$\le C\left(\int_{I_1} \left|\Delta_h^k f(x)\right|^P d\mu(x)\right)^{\frac{1}{P}} + C\left(\sum_{0\le i\le k-1} \int_{b_1-kh}^{b_1-(k-1)h} |f(x+ih)|^P\, d\mu(x)\right)^{\frac{1}{P}}$$

Using **Lemma (C),** we get

$$\left(\int_{b_1-kh+kh}^{b_1-(k-1)h+kh} |f(x)|^P\, d\mu(x)\right)^{\frac{1}{P}} \le C\left(\int_{I_1} \left|\Delta_h^k f(x)\right|^P d\mu(x)\right)^{\frac{1}{P}} + C\left(\sum_{0\le i\le k-1} \int_{b_1-kh}^{b_1-(k-1)h} |f(x)|^P\, d\mu(x)\right)^{\frac{1}{P}}$$

$$\left(\int_{b_1}^{b_1+h} |f(x)|^P\, d\mu(x)\right)^{\frac{1}{P}} \le C\left(\int_{I_1} \left|\Delta_h^k f(x)\right|^P d\mu(x)\right)^{\frac{1}{P}} + C\left(\sum_{0\le i\le k-1} \int_{b_1-(k-i)h}^{b_1-(k-1)h+ih} |f(x)|^P\, d\mu(x)\right)^{\frac{1}{P}}$$

Since $[b_1, b_1+h] \subset I_1$ and $[b_1-(k-i)h, b_1-(k-1)h+ih] \subset J$ then

$$\|f(.)\|_{P,I_1,\mu} \le C\omega_k(f,h,I_1)_{P,\mu} + C\|f(.)\|_{P,J,\mu}$$

In the same way we get the same result when the interval $I_1 = [a_1-h, b_1]$

Corollary (5.2):

Let $f \in L_{P,\mu}(I)$, $1 \le P < \infty$, let$J = [c,d] \subset I$

Let $I_1 \subset I$ be such that $J \subset I_1$ then

$$\|f(.)\|_{P,I_1,\mu} \le C\omega_k(f,|I_1|,I_1)_{P,\mu} + C\|f(.)\|_{P,J,\mu} \qquad \textbf{(5.2.1)}$$

Where C depends on the ratio $|I_1|/|J|$.

Lemma 5.3:

Let J_i be subset of ℓ_i and$f \in L_{p,\mu}(\ell_i) \cap \Delta^0(\ell_i)$, $1 \le p < \infty$,and there exists a polynomial $p_{k-1}(f) \in \prod_{k-1} \cap \Delta^0(\ell_i)$ interpolate f at the point $k-1$ inside J_i

Then for any constant $d > 0$, we have two cases.

Case(1): For $\theta_1 = \frac{j_i + j_i^{(k-1)}}{k-1} + d|J_i| < j_i^{(k-1)}$,

$$\|P_{k-1}(f)\|_{L_{p,\mu}[\frac{j_i+j_i^{(z)}}{k-1},\theta_1]} \le c(p,d)\|f\|_{L_{p,\mu}[\frac{j_i+j_i^{(z)}}{k-1},\theta_1]}$$

Case (2): For $\theta_2 = \frac{j_i + j_i^{(z)}}{k-1} - d|J_i| > j_i^{(z)}$,

$$\|P_{k-1}(f)\|_{L_{p,\mu}[\theta_2,\frac{j_i+j_i^{(k-1)}}{k-1}]} \le c(p,d)\|f\|_{L_{p,\mu}[\theta_2,\frac{j_i+j_i^{(k-1)}}{k-1}]}$$

Proof cases (1):

For $J_i \subseteq \ell_i$, and let $P_{k-1}(f) = \sum_{i=1}^{r} f(j_i) \prod_{i=1}^{r}(t_j - t_i)$, $0 \le j \le n$

be a linear polynomial of degree $\le k-1$ interpolating f inside J_i and belong to$\Delta^0(\ell_i)$ Since $f(j_i) \ge 0$,$\forall_i = 1, \dots, r$,and that $P_{k-1}(f)$ is nondecreasing for $j_i > t_j$, and hence $P_{k-1}(f) \ge 0$ for $j_i > t_j$ [since $f(j_i) \ge 0$]

Thus $f - p_{k-1}(f) \ge 0$ changes sign in side ℓ_i.

In particular $f - p_{k-1}(f) \ge 0$

for

$j_i^{(k-1)} > \frac{j_i + j_i^{(k-1)}}{k-1}$, hence $p_{k-1}(f) \le f$

for $\frac{j_i + j_i^{(k-1)}}{k-1} < \frac{j_i + j_i^{(k-1)}}{k-1} + d|J_i| < j_i^{(k-1)}$,

then for any constant $d > 0$ such that :

$\theta_1 = \frac{j_i + j_i^{(k-1)}}{k-1} + d|J_i| < j_i^{(k-1)}$, and from **proposition (B)** ,we get

$$\|P_{k-1}(f)\|_{L_{p,\mu}[\frac{j_i + j_i^{(k-1)}}{k-1}, \theta_1]} = \left(\int_{[\frac{j_i + j_i^{(k-1)}}{k-1}, \theta_1]} |P_{k-1}(f)|^p \, d\mu(x) \right)^{\frac{1}{p}}$$

$$\le c(p,d) \left(\int_{[\frac{j_i + j_i^{(k-1)}}{k-1}, \theta_1]} |f|^p \, d\mu(x) \right)^{\frac{1}{p}}$$

$$= c(p,d) \|(f)\|_{L_{p,\mu}[\frac{j_i + j_i^{(k-1)}}{k-1}, \theta_1]}$$

Since $|J_i|$ *and* $\left\| \left[\frac{j_i + j_i^{(k-1)}}{k-1}, \theta_1 \right] \right\|$ are equivalence then we get

$$\left(\int_{[\frac{j_i + j_i^{(z)}}{k-1}, \theta_1]} |P_{k-1}(f)|^p \, d\mu(x) \right)^{\frac{1}{p}} \le c(p,d) \left(\int_{[\frac{j_i + j_i^{(z)}}{k-1}, \theta_1]} |f|^p \, d\mu(x) \right)^{\frac{1}{p}}$$

Hence

$$\|P_{k-1}(f)\|_{L_{p,\mu}[\frac{j_i + j_i^{(z)}}{k-1}, \theta_1]} \le c(p,d) \|f\|_{L_{p,\mu}[\frac{j_i + j_i^{(z)}}{k-1}, \theta_1]} \quad \textbf{(5.4)}$$

Case (2):

By the same method in case (1) and in particular

$$f - p_{k-1}(f) \ge 0$$

for $j_i^{(z)} < \frac{j_i + j_i^{(z)}}{k-1}$, hence $p_{k-1}(f) \le f$ for $j_i^{(z)} < \frac{j_i + j_i^{(z)}}{k-1} + d|J_i| < \frac{j_i + j_i^{(z)}}{k-1}$,then for any constant $d > 0$,such that

$\theta_2 = \frac{j_i + j_i^{(z)}}{k-1} - d|J_i| > j_i^{(z)}$ we get

$$\|P_{k-1}(f)\|_{L_{p,\mu}[\theta_2, \frac{j_i + j_i^{(z)}}{k-1}]} = \left(\int_{[\theta_2, \frac{j_i + j_i^{(z)}}{k-1}]} |P_{k-1}(f)|^p \, d\mu(x) \right)^{\frac{1}{p}}$$

$$\le c(p,d) \left(\int_{[\theta_2, \frac{j_i + j_i^{(z)}}{k-1}]} |f|^p \, d\mu(x) \right)^{\frac{1}{p}} = c(p,d) \|f\|_{L_{p,\mu}[\theta_2, \frac{j_i + j_i^{(z)}}{k-1}]}$$

Since $|J_i|$ *and* $\left\| \left[\theta_2, \frac{j_i + j_i^{(z)}}{k-1} \right] \right\|$ are equivalence then we get

$$\left(\int_{[\theta_2, \frac{j_i + j_i^{(k-1)}}{k-1}]} |P_{k-1}(f)|^p \, d\mu(x) \right)^{\frac{1}{p}} \le c(p,d) \left(\int_{[\theta_2, \frac{j_i + j_i^{(k-1)}}{k-1}]} |f|^p \, d\mu(x) \right)^{\frac{1}{p}}$$

Hence

$$\|P_{k-1}(f)\|_{L_{p,\mu}[\theta_2, \frac{j_i + j_i^{(k-1)}}{k-1}]} \le c(p,d) \|f\|_{L_{p,\mu}[\theta_2, \frac{j_i + j_i^{(k-1)}}{k-1}]} \quad \textbf{(5.5)}$$

From **lemma (5.3)** and since J_A super in the interpolate set of J_i and by case (1) and case (2) we get

$\|P_{k-1}(f)\|_{L_{p,\mu}(J_A)} \le c(p,d) \|f\|_{L_{p,\mu}(J_A)}$

Lemma 5.6:

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For $f \in L_{p,\mu}(\ell_i) \cap \Delta^0(\ell_i)$, then there exists$p_{k-1}(f) \in \prod_{k-1} \cap \Delta^0(\ell_i)$ interpolate f at $k-1$ points inside ℓ_i, such that
$\|f - P_{k-1}(f)\|_{L_{p,\mu}(\ell_i)} \leq C(p,k)\omega_k^{\varphi}(f, |\ell_i|, \ell_i)_{p,\mu}$ **(5.7)**

Proof:

For an interval J_i , such that
$J_i = \left[\frac{j_i + j_i^{(z)}}{k-1}, \frac{j_i + j_i^{(k-1)}}{k-1}\right]$, we have $|J_i| = \frac{j_i^{(k-1)} - j_i^{(z)}}{k-1}$,

We define $\ell_i / J_i = \left(\left[j_i^{(z)}, \frac{j_i + j_i^{(z)}}{k-1}\right) \cup \left(\frac{j_i + j_i^{(k-1)}}{k-1}, j_i^{(k-1)}\right]\right)$.

Since $\ell_i = (k-1)J_i$, that is ℓ_i consists of $k-1$ interval J_i with $(k-1)|J_i| = |\ell_i|, k \geq 4$, let
$|J_i| \approx \left|\left[j_i^{(z)}, \frac{j_i + j_i^{(z)}}{k-1}\right)\right|$ and $|J_i| \approx \left|\left(\frac{j_i + j_i^{(k-1)}}{k-1}, j_i^{(k-1)}\right]\right|$ **(5.8)**

From **(4.1)** suppose that $J_i \subset (k-1)J_i = \ell_i, k \geq 4$

Since $f \in L_{p,\mu}(\ell_i) \cap \Delta^0(\ell_i)$, then by **lemma (5.3)** there exists$p_{k-1}(f)$interpolate f at k points inside J_i, hence from
$\|P_{k-1}(f)\|_{L_{p,\mu}(J_i)} \leq c(p,d)\|f\|_{L_{p,\mu}(J_i)}$

And since **(5.8)** are satisfy then we get, where
$J_i' = \left[j_i^{(z)}, \frac{j_i + j_i^{(z)}}{k-1}\right)$ and $J_i'' = \left(\frac{j_i + j_i^{(k-1)}}{k-1}, j_i^{(k-1)}\right]$, that

$$\|P_{k-1}(f)\|_{L_{p,\mu}(J_i')} = \left(\int_{J_i'} |P_{k-1}(f)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

Suppose that $Q \in \prod_{k-1}$, Q interpolates f at $k-1$ points insideJ_i'

Then $P_{k-1}(f) = P_{k-1}(Q)$

We take the integral for both sided, we get

$$\|P_{k-1}(f)\|_{L_{p,\mu}(J_i')} = \left(\int_{J_i'} |P_{k-1}(f)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$=\left(\int_{J_i'} |P_{k-1}(Q)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

By **Proposition (B)** we get

$$\left(\int_{J_i'} |P_{k-1}(f)|^p \, d\mu(x)\right)^{\frac{1}{p}} \leq \mathrm{C(p,d)}\left(\int_{J_i'} |Q(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

Since $Q = Q(x) - f(x) + f(x)$, then

$$= \mathrm{C(p,d)}\left(\int_{J_i'} |Q(x) - f(x) + f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq \mathrm{C(p,d)}\left(\int_{J_i'} |Q(x) - f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} + \mathrm{C(p,d)}\left(\int_{J_i'} |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq \mathrm{C(p,d)}\|Q(.) - f(.)\|_{p,\mu} + \mathrm{C(p,d)}\left(\int_{J_i'} |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

By **(3.1)**$\omega_k(f,x,I)_{p,\mu} = sup\left\|\Delta_h^k(f,.,I)\right\|_{p,\mu}$, $p \in [1,\infty)$ we get

$$\left(\int_{J_i'} |P_{k-1}(f)|^p \, d\mu(x)\right)^{\frac{1}{p}} \leq \mathrm{C(p,d)}\omega_k(f, |J_i'|, J_i')_{p,\mu} + \mathrm{C(p,d)}\left(\int_{J_i'} |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

by using **(5.1.1)** we get
$\|P_{k-1}(f)\|_{L_{p,\mu}(J_i')} \leq C(p,d)\|f\|_{L_{p,\mu}(J_i')}$, and also

$$\|P_{k-1}(f)\|_{L_{p,\mu}(J_i'')} \leq C(p,d)\|f\|_{L_{p,\mu}(J_i'')}$$

And from the fact $\ell_i / J_i = \left(\left[j_i^{(z)}, \frac{j_i + j_i^{(z)}}{k-1}\right) \cup \left(\frac{j_i + j_i^{(k-1)}}{k-1}, j_i^{(k-1)}\right]\right)$ we have

$$\|P_{k-1}(f)\|_{L_{p,\mu}(\ell_i / J_i)} \leq C(p,d)\|f\|_{L_{p,\mu}(\ell_i / J_i)}$$

Now applied the same relation in **(4.2)** for an interval J_i, we get

$$\|f-P_{k-1}(f)\|_{L_{p,\mu}(J_i)} = \left(\int_{J_i} |f-P_{k-1}(f)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$=\left(\int_{J_i} |f-g^*+g^*-P_{k-1}(f)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{J_i} |f-g^*|^p \, d\mu(x)\right)^{\frac{1}{p}} + \left(\int_{J_i} |g^*-P_{k-1}(f)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{J_i} |f-g^*|^p \, d\mu(x)\right)^{\frac{1}{p}} + C(p)\left(\int_{J_i} |P_{k-1}(g^*-f)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq \|f-g^*\|_{L_{p,\mu}(J_i)} + C(p)\|g^*-f\|_{L_{p,\mu}(J_i)}$$

$$= C(p)\omega_k(f,|J_i|,J_i)_{p,\mu}$$

Hence we get

$$\|f-P_{k-1}(f)\|_{L_{p,\mu}(J_i)} \leq C(p)\omega_k(f,|J_i|,J_i)_{p,\mu}$$

Since $|J_i| \to 0$, then we get

$$\|f-P_{k-1}(f)\|_{L_{p,\mu}(J_i)} \leq C(p)\omega_k^{\varphi}(f,|J_i|,J_i)_{p,\mu}$$

Since $J_i \subset (k-1)J_i = \ell_i$then we get

$$\|f-P_{k-1}(f)\|_{L_{p,\mu}(\ell_i)} \leq C(p,k)\omega_k^{\varphi}(f,|\ell_i|,\ell_i)_{p,\mu}$$

Lemma 5.9:

For $f \in L_{p,\mu}(\ell_i) \cap \Delta^0(\ell_i)$, $1 \leq p < \infty$, then there exists$q_{k-1}(f) \in \prod_{k-1} \cap \Delta^0(\ell_i)$ interpolate f at $k-1$ points inside ℓ_i, such that

$$\|f-q_{k-1}(f)\|_{L_{p,\mu}(\ell_i)} \leq C(p,k)\tau_k(f,|\ell_i|,\ell_i)_{p,\mu}$$

Proof:

From **Lemma (5.6),** there exist a polynomial q_{k-1} of degree $\leq k-1$ copositive with $f$in ℓ_i and interpolate $f$at the points inside ℓ_i, hence from **Lemma (5.6)** we get

$$\|f-q_{k-1}(f)\|_{L_{p,\mu}(\ell_i)} \leq C(p,k)\omega_k^{\varphi}(f,|\ell_i|,\ell_i)_{p,\mu}$$

$$|f-q_{k-1}| \leq C(p)\left(\int_{\ell_i} |f-q_{k-1}|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$=C(p)\|f-q_{k-1}(f)\|_{L_{p,\mu}(\ell_i)}$$

Then we get

$$|f-q_{k-1}| \leq C(p,k)\omega_k^{\varphi}(f,|\ell_i|,\ell_i)_{p,\mu}$$

Now by taking the $L_{p,\mu}(\ell_i)$ -norm for both sides we get

$$\|f-q_{k-1}(f)\|_{L_{p,\mu}(\ell_i)} \leq C(p,k)\left\|\omega_k^{\varphi}(f,|\ell_i|,\ell_i)\right\|_{L_{p,\mu}(\ell_i)}$$

By τ -modulus (or Sendove Popov modulus) with measure μ, for f on ℓ_i, we get

$$\|f-q_{k-1}(f)\|_{L_{p,\mu}(\ell_i)} \leq C(p,k)\tau_k(f,|\ell_i|,\ell_i)_{p,\mu}$$

6-*Main Results:*

Theorem 6.1:

For $f \in L_{p,\mu}(I) \cap \Delta^0(J_r)$,$1 \leq p < \infty$ and let $g_{k-1}(f) \in \prod_{k-1} \cap \Delta^0(J_r)$, $k > 1$ interpolate f at $k-1$ points in side J_A where $J_A = [a+d|I|, b-d|I|]$, then

$$\|f-g_{k-1}(f)\|_{L_{p,\mu}(I)} \leq C(p,k)\omega_k^{\varphi}(f,|I|,I)_{p,\mu}$$

Proof:

Let $d > 0$ be a fixed and let ℓ_i , $i = 1,\dots,r$ be an interval of length $|\ell_i| = j_i^{(k-1)} - j_i^{(z)}$, $k > 1$, $z = 1,\dots,k-2$ in the center of $I = [a,b]$, that is $\mathrm{dis}(\ell_i, a) = \mathrm{dis}(\ell_i, b)$ then by **Lemma(5.6)** there exist linear apolynomial$q_{k-1}^*(f) \in \prod_{k-1}$ of best approximation, copositive and interpolate f at k points in side $\ell_i \cap J_A$, hence we get

$$\| f-q_{k-1}^*\|_{L_{p,\mu}(I)} \leq C(p,k)\omega_k^{\varphi}(f,|I|,I)_{p,\mu} \qquad \textbf{(6.2)}$$

Also by **lemma (5.6)** there exist a linear polynomial $h_{k-1} = h_{k-1}(f) \in \prod_{k-1}$ of best approximation, copositive and interpolate f at k points in side $J_B = \left[b - d|I|, b - \frac{1}{2}d|I|\right]$,$d < \frac{1}{2}$ and $b - \frac{1}{2}d|I| \leq b$hence

$$\|f - h_{k-1}(f)\|_{L_{p,\mu}(I)} = \left(\int_I |f - h_{k-1}(f)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$= \left(\int_I |f - q_{k-1}^* + q_{k-1}^* - h_{k-1}(f)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq C(p)\left(\int_I |f - q_{k-1}^*|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$+C(p)\left(\int_I | \quad |q_{k-1}^* - h_{k-1}(f)^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq C(p)\left(\int_I |f - q_{k-1}^*|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$+C(p)\left(\int_I |h_{k-1}(f - q_{k-1}^*)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq C(p)\left(\int_I |f - q_{k-1}^*|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$+C(p)\left(\int_{J_c} |h_{k-1}(f - q_{k-1}^*)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

Where $J_c = [b - d|I|, b]$ and $|J_B| \approx |J_c|$
since we have from an interval J_B and J_c that $b - d|I| \leq \mathrm{b} - \frac{1}{2}\mathrm{d}|I|$ hence

$$\left(\int_I |f - h_{k-1}(f)|^p \, d\mu(x)\right)^{\frac{1}{p}} \leq C(p)\left(\int_I |f - q_{k-1}^*|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$+C(p)\left(\int_{J_B} |h_{k-1}(f - q_{k-1}^*)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

Then by **lemma (5.3)** we get

$$\left(\int_I |f - h_{k-1}(f)|^p \, d\mu(x)\right)^{\frac{1}{p}} \leq C(p)\left(\int_I |f - q_{k-1}^*|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$+C(p)\left(\int_{J_B} | f - q_{k-1}^*|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

That is

$$\|f - h_{k-1}(f)\|_{L_{p,\mu}(I)} \leq C(p)\|f - q_{k-1}^*\|_{L_{p,\mu}(I)} + C(p)\|f - q_{k-1}^*\|_{L_{p,\mu}(J_B)}$$

Then by **lemma (5.9)** and inequality **(6.2)** we get

$$\left(\int_I |f - h_{k-1}(f)|^p \, d\mu(x)\right)^{\frac{1}{p}} \leq C(p,k)\omega_k^{\varphi}(f, |I|, I)_{p,\mu}$$

Therefore

$$\|f - h_{k-1}(f)\|_{L_{p,\mu}(I)} \leq C(p,k)\omega_k^{\varphi}(f, |I|, I)_{p,\mu} \qquad \textbf{(6.3)}$$

Also there exist a linear polynomial $g_{k-1}(f) \in \prod_{k-1}$,copositive and interpolate f at k points in side J_A where $a + d|I| \geq a$,
also $b - d|I| \leq b$ hence

$$\|f - g_{k-1}\|_{L_{p,\mu}(I)} = \|f - h_{k-1}(f) + h_{k-1}(f) - g_{k-1}(f)\|_{L_{p,\mu}(I)}$$

Hence

$$\|f-g_{k-1}\|_{L_{p,\mu}(I)}=\left(\int_I |f-g_{k-1}(f)|^p\,d\mu(x)\right)^{\frac{1}{p}}=\left(\int_I |f-h_{k-1}(f)+h_{k-1}(f)-g_{k-1}(f)|^p\,d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq C(p)\left(\int_I |f-h_{k-1}|^p\,d\mu(x)\right)^{\frac{1}{p}}+C(p)\left(\int_I |g_{k-1}-h_{k-1}(f)|^p\,d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq C(p)\left(\int_I |f-h_{k-1}|^p\,d\mu(x)\right)^{\frac{1}{p}}+C(p)\left(\int_I |g_{k-1}(f-h_{k-1})|^p\,d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq C(p)\left(\int_I |f-h_{k-1}|^p\,d\mu(x)\right)^{\frac{1}{p}}+C(p)\left(\int_{J_k} |g_{k-1}(f-h_{k-1})|^p\,d\mu(x)\right)^{\frac{1}{p}}$$

Where $J_k=[a+d|I|,b]$, and $|J_A|\approx|J_k|$, since we have from an interval J_A and J_k that $a+d|I|\leq b-d|I|$, hence

$$\left(\int_I |f-g_{k-1}(f)|^p\,d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq C(p)\left(\int_I |f-h_{k-1}|^p\,d\mu(x)\right)^{\frac{1}{p}}+C(p)\left(\int_{J_A} |g_{k-1}(f-h_{k-1})|^p\,d\mu(x)\right)^{\frac{1}{p}}$$

Then by **lemma (5.3)**, we get

$$\left(\int_I |f-g_{k-1}(f)|^p\,d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq C(p)\left(\int_I |f-h_{k-1}|^p\,d\mu(x)\right)^{\frac{1}{p}}+C(p)\left(\int_{J_A} |f-h_{k-1}|^p\,d\mu(x)\right)^{\frac{1}{p}}$$

Then by **lemma (5.9)** and inequality **(6.3)** we get

$$\left(\int_I |f-g_{k-1}|^p\,d\mu(x)\right)^{\frac{1}{p}}\leq C(p,k)\omega_k^{\varphi}(f,|I|,I)_{p,\mu}$$

Therefore

$$\|f-g_{k-1}(f)\|_{L_{p,\mu}(I)}\leq C(p,k)\omega_k^{\varphi}(f,|I|,I)_{p,\mu} \qquad \textbf{(6.4)}$$

(6.4) means there exist a polynomial copositive and interpolate f in an interval J_A , such that $a+d|I|\leq b$ will satisfy the Whitney theorem .

And by the same method in the above we can get the same result for an interval J_A such that $a\leq b-d|I|$.

Hence the result is true for I. If $d=0$ then the inequality **(6.4)** is not true.

Theorem 6.5:

For $f\in L_{p,\mu}(I)\cap\Delta^0(J_r)$, $1\leq p<\infty$ then there exist a polynomial $p_{k-1}(f)\in\prod_{k-1}\cap\Delta^0(J_r)$, $k>1$, *such that*

$$\|f-p_{k-1}\|_{L_{p,\mu}(I)}\leq C(p,k)\tau_k(f,|I|,I)_{p,\mu}$$

Proof:

By **lemma (5.6)** there exists polynomial $g^*\in\prod_{k-1}\cap\Delta^0(J_r)$ of degree $\leq k-1$ and g^*best approximation to f on $I=[a,b]$

$$\|f-p_{k-1}(f)\|_{L_{p,\mu}(I)}=\|f-g^*+g^*-p_{k-1}(f)\|_{L_{p,\mu}(I)}$$

Hence

$$\|f-p_{k-1}(f)\|_{L_{p,\mu}(I)}=\left(\int_I |f-g^*+g^*-p_{k-1}(f)|^p\,d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq \left(\int_I |f - g^*|^p \, d\mu(x)\right)^{\frac{1}{p}} + \left(\int_I | g^* - p_{k-1}(f) |^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq \left(\int_I |f - g^*|^p \, d\mu(x)\right)^{\frac{1}{p}} + C(p)\left(\int_I |p_{k-1}(g^* - f) |^p \, d\mu(x)\right)^{\frac{1}{p}}$$

$$= \|f - g^*\|_{L_{p,\mu}(I}$$

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