Australian Journal of Basic and Applied Sciences, 10(8) April 2016, Pages: 25-30

AUSTRALIAN JOURNAL OF BASIC AND APPLIED SCIENCES

ISSN:1991-8178 EISSN: 2309-8414
Journal home page: www.ajbasweb.com

Controllability of Fractional Control Systems Using Schauder Fixed Point Theorem

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ARTICLE INFO

Article history:
Received 12 February 2016
Accepted 29 March 2016
Available online 4 April 2016

Keywords:
controllability, fractional control system, Schauder theorem, Caputo fractional derivative.

ABSTRACT

In this paper, sufficient conditions for the controllability of nonlinear fractional control systems are established, by using some techniques of nonlinear functional analysis, such as, Schauder's fixed point theorem. Example is provided to illustrate the theorem.

INTRODUCTION

Dynamical systems represented by ordinary differential equations are extensively studied in the literature. Many mathematical problems in science and engineering are represented by fractional differential equations, thus these kinds of equations have increasingly attracted the attention of many researches (Ahmeda, E., A. Elgazzar, 2007; Luchko, Y., 2010). In fact a fractional differential equations is considered as an alternative model to a nonlinear differential equations (Bonilla, B., 2007). Theory of fractional differential equations has been extensively studied by many authors (Balachandran, K., J. Trujillo, 2010; Kilbas, A., 2006). One of the basic qualitative behaviours of a dynamical system is stability. This problem has been discussed for fractional differential equations in (Bonnet, C., J. Partington, 2000). Besides the stability problem, another most important qualitative behaviour of dynamical system is controllability. This means that it is possible to steer any initial state of the system to any final state in some finite time using an admissible control.

The concept of controllability for linear or nonlinear systems which are represented by ordinary differential equations or partial differential equations has been studied in (Klamka, J., 1993; Balachandran, K., J. Dauer, 1987). So it is natural to study the concept of controllability for dynamic systems represented by fractional differential equations. Controllability of fractional dynamic systems in finite dimensional space is discussed in (Chen, Y., 2006; Monje, C., 2010).

The aim of this paper is to study the controllability of nonlinear fractional dynamic systems in general from by using the Schauder fixed point theorem. The rest of this paper is organized as follows. In Section 2, we present some necessary definitions and preliminary results that will be used to prove our main result. The proof of our main result is given in Section 3. Finally, an example is included in Sections 4.

1. Preliminaries:

In this section, some well known fractional operators, special function, definitions and theorems that will be used in this paper have been presented (Kilbas, A., 2006; Chen, Y., 2006; Samko, S., 1993; Kreyszig, E., 1978; Balachandran, K., 2012).

To Cite This Article: Naseif J. Al-Jawari and Sabah Mahmood Shaker., Controllability of Fractional Control Systems Using Schauder Fixed Point Theorem. ***Aust. J. Basic & Appl. Sci.,*** *10(8): 25-30, 2016*

Let $p, q > 0$, with n-1 $< p < n$, n-1 $< q < n$, and $n \in$ N, [a,b] $\subset$ R, D the usual differential operator, and let h be a suitable real function (for example, in general, it is sufficient to have $h \in L_1$(a,b)). Let R^m be the m-dimensional Euclidean space.

The following definitions and properties are well known, for $p, q > 0$ and h as a suitable function:

(a) Riemann-Liouville fractional operators (RLFO):

(i) The left RLFO:

$(\mathrm{I}_{a+}^{p} h)(x) = \frac{1}{\Gamma(p)} \int_a^x (x-t)^{p-1} h(t)\, dt \quad (x > a)$ **(1)**

$(D_{a+}^{p} h)(x) = \mathrm{D}^n (\mathrm{I}_{a+}^{n-p} h)(x), \quad (x > a)$

(ii) The right RLFO:

$(\mathrm{I}_{b-}^{p} h)(x) = \frac{1}{\Gamma(p)} \int_x^b (t-x)^{p-1} h(t)\, dt \quad (x < b)$ **(2)**

$(D_{b-}^{p} h)(x) = \mathrm{D}^n (\mathrm{I}_{b-}^{n-p} h)(x), \quad (x < b).$

(b) The Caputo fractional derivative :

$({}^{c}D_{a+}^{p} h)(x) = (\mathrm{I}_{a+}^{n-p} \mathrm{D}^n h)(x), \quad (x > a),$ **(3)**

and in particular $\mathrm{I}_{0+}^{p}\, {}^{c}D_{0+}^{p} h(t) = h(t) - h(0) \quad (0 < p < 1)$.

The following relation is well known:

$(D_{a+}^{p} h)(x) = ({}^{c}D_{a+}^{p} h)(x) + \sum_{j=0}^{n-1} \frac{h^{(j)}(a)}{\Gamma(1+j-p)} (x-a)^{j-p}.$

Thus, we get that

$({}^{c}D_{a+}^{p} 1) = 0$, and $(D_{a+}^{p} 1) = \frac{(x-a)^{-p}}{\Gamma(1-p)}$.

The following properties of the above operators:

1. $({}^{c}D_{a+}^{p} 1) = 0.$
2. $(D_{a+}^{p} 1) = \frac{(x-a)^{-p}}{\Gamma(1-p)}.$
3. $\mathrm{I}_{a+}^{p} (h(t) + g(t)) = \mathrm{I}_{a+}^{p} h(t) + \mathrm{I}_{a+}^{p} g(t).$
4. $\mathrm{I}_{a+}^{p} \mathrm{I}_{a+}^{q} h(t) = \mathrm{I}_{a+}^{p+q} h(t) = \mathrm{I}_{a+}^{q} \mathrm{I}_{a+}^{p} h(t).$
5. $D_{a+}^{p} \mathrm{I}_{a+}^{p} h(t) = h(t).$
6. $\mathrm{I}_{a+}^{p} D_{a+}^{p} h(t) = h(t) - \frac{(\mathrm{I}_{a+}^{1-p} h)(a)}{\Gamma(p)} (x-a)^{p-1} \quad (0 < p < 1).$
7. $\mathrm{I}_{a+}^{p}\, {}^{c}D_{a+}^{p} h(t) = h(t) - h(a), \quad 0 < p < 1.$
8. In general, $D_{a+}^{p} D_{a+}^{q} h(t) \neq D_{a+}^{p+q} h(t)$, and $D_{a+}^{p} D_{0+}^{q} h(t) \neq D_{a+}^{q} D_{0+}^{p} h(t)$.

From all the above we see that, in general, both the Riemann-Liouville and the Caputo fractional operators possess neither semigroup nor commutative, which are inherent to the derivatives to integer order. However, with some restrictions, for example $0 < p < 1$ and if h is a continuous function in [a,b], both properties hold true for both of the aforementioned operators.

The Caputo fractional derivative is more often used in applied research. For brevity of notation let us take I_{0+}^{p} as I^p and ${}^{c}D_{0+}^{p}$ as D^p.

Here consider the well known Mittag-Leffler function

$E_{p,q}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kp+q)}$, for $p, q > 0$ **(4)**

For q = 1,

$E_{p,1}(\lambda z^p) = E_p(\lambda z^p) = \sum_{k=0}^{\infty} \frac{\lambda^k z^{kp}}{\Gamma(pk+1)}$, $\lambda, z \in \mathbb{C}$

has the interesting property ${}^{c}D_{0+}^{p}\ E_p(\lambda t^p) = \lambda\, E_p(\lambda t^p)$.

Now, consider the linear dynamical system represented by a fractional differential equation of the form

$D^p x(t) = Ax(t) + Bu(t)$, $t \in J = [0,b]$,

$x(0) = x_0$, **(5)**

with $0 < p < 1$, $x \in R^n$, $u \in R^m$, A is an nxn matrix and B is an nxm matrix.

The solution of the system **(5)** given by the following expression (6):

$x(t) = E_p(At^p)x_0 + \int_o^t (t-s)^{p-1} E_{p,p}(A(t-s)^p)Bu(s)ds$ **(6)**

where $E_p(At^p)$ is the matrix extension of the aforementioned Mittag-Leffler function with the following representation:

$E_p(At^p) = \sum_{k=0}^{\infty} \frac{A^k t^{kp}}{\Gamma(1+kp)}$

with the property ${}^{c}D_{0+}^{p}\ E_p(At^p) = A\, E_p(At^p)$.

In particular the solution **(6)** satisfies the fractional differential equation **(5)**. Throughout this paper, the fractional derivative is taken in the Caputo sense.

Definition 1 (Chen, Y., 2006):

The system **(5)** is said to be *controllable* on J if for every $x_{0,}\, x_1 \in R^n$ there exists a control $u(t)$such that the solution $x(t)$of such a system satisfies the conditions $x(0) = x_0$ and $x(b) = x_1$.

Define the *controllability Grammian matrix W* as:

$$W = \int_0^b (b-s)^{p-1}\,[E_{p,p}(A(b-s)^p B]\,[E_{p,p}(A(b-s)^p)B]^* ds. \tag{7}$$

Here $*$ denotes the matrix transpose.

Theorem 1:

The linear control system **(5)** is *controllable* on $[0,b]$ if and only if the *controllability Grammian matrix* **(7)** is positive definite, for some $b > 0$.

Definition 2:

A subset E of $C([a,b])$ is said to be *equicontinuous*, if for each $\epsilon > 0$ there is a $\delta > 0$, depending only on ϵ, such that for all $f \in E$ and all $x, y \in [a,b]$ satisfying $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Note that δ does not depend on *f*.

Theorem 2:

(Schauder Theorem) Every continuous operator that maps a closed convex subset of a Banach space into a compact subset of itself has at least one fixed point.

2. Main Result:

In this section we prove the main result that deals with the controllability of nonlinear fractional control system.

Consider the nonlinear fractional dynamic system represented by a fractional differential equation of the form

$$D^p x(t) = Ax(t) + Bu(t) + Q(t, F(t, x(t))), t \in J = [0, b],$$
$$x(0) = x_0, \tag{8}$$

where $0 < p < 1$, $x \in R^n$, $u \in R^m$, A is an nxn matrix and B is an nxm matrix. The nonlinear operators $F \in C(JxR^n, R^n), Q \in C(JxR^n, R^n)$ are all uniformly bounded continuous operators.

Let $C_n(J)$ be a Banach space of continuous R^n valued functions defined on the interval J with the supremum norm (i.e., for $x \in R^n$, $\|x\| = \sup\{|x(t)| : t \in J\}$).

For each $z \in C_n(J)$, consider the linear fractional dynamic system

$$D^p x(t) = Ax(t) + Bu(t) + Q(t, F(t, z(t))),$$
$$x(0) = x_0,$$

Then the solution is given by the form

$$x(t) = E_p(At^p)x_0 + \int_o^t (t-s)^{p-1} E_{p,p}(A(t-s)^p)Bu(s)ds$$
$$+ \int_o^t (t-s)^{p-1} E_{p,p}(A(t-s)^p)\, Q(s, F(s, z(s)))ds. \tag{9}$$

So according to the solution in **(9)** the following definition has been presented.

Definition 3:

The system **(8)** is said to be *controllable* on J if for every $x_0, x_1 \in C_n(J)$ there exists a control $u(t)$ such that the solution $x(t)$ in **(9)** satisfies the conditions $x(0) = x_0$ and $x(b) = x_1$.

Now, let $c_1 = sup\|E_{p,p}(A(b-t)^p\|$, $c_2 = sup\|E_p(At^p)x_0\|$, and assume the following condition:

(I) $F : JxR^n \longrightarrow R^n$ and $Q : JxR^n \longrightarrow R^n$ are uniformly continuous and there exists a positive constant $c_3 > 0$ such that $\|Q(t, F(t, z(t)))\| \le c_3$ for all $t \in J, z \in C_n(J)$.

Theorem 3:

If the linear system **(5)** is controllable and the hypothesis **(I)** holds, then the system **(8)** is controllable on J.

Proof :

Define the operator $O : C_n(J) \longrightarrow C_n(J)$ by

$$O(z)(t) = E_p(At^p)x_0 + \int_o^t (t-s)^{p-1}\, E_{p,p}(A(t-s)^p)Bu(s)ds +$$
$$+ \int_o^t (t-s)^{p-1} E_{p,p}(A(t-s)^p)\, Q(s, F(s, z(s)))ds, \tag{10}$$

where the control $u(t)$ is defined as follows

$u(t) = B^* E_{p,p}(A^*(b-t)^p) W^{-1}[x_1 - E_p(Ab^p)x_0 -$
$- \int_o^b (b-s)^{p-1} E_{p,p}(A(b-s)^p)\, Q(s, F(s, z(s)))ds]$. **(11)**

Now, by using the Schauder fixed point theorem (Theorem 2) we want to prove that the operator O in **(10)** has a fixed point.

Therefore, define a closed convex subset
$\Upsilon(r) = \{z \in C_n(J): \|z\| \le r\}$,
where r is a positive constant given by
$r = c_2 + \frac{c_1 b^p \|B\| k}{p} + \frac{c_1 b^p c_3}{p}$,
And
$k = c_1 \|B^*\| \|W^{-1}\| [|x_1| + c_2 + \frac{c_1 b^p c_3}{p}]$.

Then by taken the norm of both sides of **(11)** we get that
$\|u(t)\| \le \|B^*\| \, \|E_{p,p}(A^*(b-t)^p)\| \, \|W^{-1}\| [|x_1| + \|E_p(Ab^p)x_0\| +$
$+ \int_o^b (b-s)^{p-1} \| E_{p,p}(A(b-s)^p)\| \; \|Q(s, F(s, z(s)))\| ds]$
$\le c_1 \|B^*\| \|W^{-1}\| [|x_1| + c_2 + \frac{c_1 b^p c_3}{p}] = k.$

Now, if we define the operator $O : C_n(J) \longrightarrow \Upsilon(r)$ as in equation **(10)** and taking the norm of both sides we obtain that
$\| O(z)(t) \| \le \|E_p(At^p)x_0\| + \int_o^t (t-s)^{p-1} \; \|E_{p,p}(A(t-s)^p)\| \, \|B\| \, \|u(s)\| ds +$
$+ \int_o^t (t-s)^{p-1} \|E_{p,p}(A(t-s)^p)\| \, \|Q(s, F(s, z(s)))\| ds$
$\le c_2 + \frac{c_1 b^p \|B\| k}{p} + \frac{c_1 b^p c_3}{p} = r.$

Since F and Q are continuous and $\|O(z)(t)\| \le r$, it follows that the operator O is also continuous and maps $\Upsilon(r)$ into itself.

Here, let us take $t_1, t_2 \in J$ with $t_1 < t_2$, and for all $z \in \Upsilon(r)$ we have to show that $O[\Upsilon(r)]$ is equicontinuous for all $r > 0$:

$\|u(t_1) - u(t_2)\| \le \|B^* E_{p,p}(A^*(b-t_1)^p) - B^* E_{p,p}(A^*(b-t_2)^p)\| \, \|W^{-1}\|$
x $[\|x_1\| + \|E_p(Ab^p)x_0\| + \int_o^b (b-s)^{p-1} \| E_{p,p}(A^*(b-s)^p)\|$
x $\|Q(s, F(s, z(s)))\| ds]$ **(12)**

Also we have
$\|x(t_1) - x(t_2)\| = \| E_p(At_1^p)x_0 - E_p(At_2^p)x_0 + \int_o^{t_1} (t_1-s)^{p-1} E_{p,p}(A(t_1-s)^p)$
$Bu(s)ds - \int_o^{t_2} (t_2-s)^{p-1} E_{p,p}(A(t_2-s)^p) Bu(s)ds + \int_o^{t_1} (t_1-s)^{p-1}$
x $E_{p,p}(A(t_1-s)^p)\, Q ds - \int_o^{t_2} (t_2-s)^{p-1} E_{p,p}(A(t_2-s)^p)\, Q ds \|$
$\le \| E_p(At_1^p)x_0 - E_p(At_2^p)x_0 \| + \int_{t_1}^{t_2} (t_2-s)^{p-1} \|E_{p,p}(A(t_2-s)^p)\|$
x $\|B\| \, \|u(s)\| ds + \int_o^{t_1} [(t_1-s)^{p-1} \|E_{p,p}(A(t_1-s)^p)\|$
$-(t_2-s)^{p-1} \|E_{p,p}(A(t_2-s)^p)] \, \|B\| \, \|u(s)\| ds + \int_o^{t_1} [(t_1-s)^{p-1}$
x $\|E_{p,p}(A(t_1-s)^p)\| - (t_2-s)^{p-1} \|E_{p,p}(A(t_2-s)^p)]$
x $\|Q\| ds + \int_{t_1}^{t_2} (t_2-s)^{p-1} \|E_{p,p}(A(t_2-s)^p)\| \, \|Q\| ds$ **(13)**

Thus the right-hand side of equations **(12)** and **(13)** is independent of $z \in \Upsilon(r)$ and tends to zero as $t_1 \longrightarrow t_2$. Hence $O[\Upsilon(r)]$ is equicontinuous for all $r > 0$ and, by the regularity assumption on F and Q, the operator is continuous, and hence it is completely continuous by the application of the Arzela-Ascoli theorem (Kreyszig, E., 1978).

Since $\Upsilon(r)$ is closed, bounded and convex, then the Schauder fixed point theorem guarantees that the operator O has a fixed point $z \in \Upsilon(r)$ such that $O(z) = z = x$ (when $x(t)$ be as in **(9)**).

By substituting **(11)** into **(10**) one can easily see that $x(b) = x_1$. Hence the system **(8)** is controllable on J.

3. Example:

In this section we shall apply the result established in Theorem 3, for the following fractional control system. For a fractional differential equations without control variables and their explicit solutions, one can see (Kilbas, A., 2006).

Now, consider the following fractional integrodifferential system:

$$D^p x(t) = Ax(t) + Bu(t) + Q(t, x(t), \int_0^t h\big(t, s, x(s)\big) ds),$$
$$x(0) = x_0, \tag{14}$$
where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $0 < p < 1$, $t \in [0, b]$ and Q is taken as follows

$$Q(t, x(t), \int_0^t h\big(t, s, x(s)\big) ds) = \begin{pmatrix} 0 \\ \frac{\int_0^t e^{-x_1(s)} ds}{1 + x_1^2(t) + x_2^2(t)} \end{pmatrix}.$$

Here the nonlinear function includes an integral term which has an interesting physical meaning in viscoelasticity (Renardy, M., W. Hrusa ., J. Nohel, 1987).
Solution.

Here $x(t) = (x_1(t),\ x_2(t))^*$ with $x_1(t) = x(t)$, $D^{\frac{p}{2}} x_1(t) = x_2(t)$.

The Mittag-Leffler matrix function is given by

$$E_p(At^p) = \begin{pmatrix} \sum_{j=0}^{\infty} \frac{(-1)^j\, t^{2jp}}{\Gamma(2jp+1)} & \sum_{j=0}^{\infty} \frac{(-1)^j\, t^{(2j+1)p}}{\Gamma((2j+1)p+1)} \\ -\sum_{j=0}^{\infty} \frac{(-1)^j\, t^{(2j+1)p}}{\Gamma((2j+1)p+1)} & \sum_{j=0}^{\infty} \frac{(-1)^j\, t^{2jp}}{\Gamma(2jp+1)} \end{pmatrix}$$

And

$$E_{p,p}(A(b-s)^p) = \begin{pmatrix} \sum_{j=0}^{\infty} \frac{(-1)^j\, (b-s)^{2jp}}{\Gamma((2j+1)p)} & \sum_{j=0}^{\infty} \frac{(-1)^j\, (b-s)^{(2j+1)p}}{\Gamma(2p(j+1))} \\ -\sum_{j=0}^{\infty} \frac{(-1)^j\, (b-s)^{(2j+1)p}}{\Gamma(2p(j+1))} & \sum_{j=0}^{\infty} \frac{(-1)^j\, (b-s)^{2jp}}{\Gamma((2j+1)p)} \end{pmatrix}$$

Suppose that

$$g_1 = \sum_{j=0}^{\infty} \frac{(-1)^j\, (b-s)^{2jp}}{\Gamma((2j+1)p)}, \quad g_2 = \sum_{j=0}^{\infty} \frac{(-1)^j\, (b-s)^{(2j+1)p}}{\Gamma(2p(j+1))}.$$

By simple matrix calculation one can see that the controllability matrix

$$W = \int_0^b (b-s)^{p-1} \big[E_{p,p}(A(b-s)^p) B\big]\big[E_{p,p}(A(b-s)^p) B\big]^* ds$$
$$= \int_0^b (b-s)^{p-1} \begin{pmatrix} g_2^2 & g_1 g_2 \\ g_1 g_2 & g_1^2 \end{pmatrix} ds$$

is positive defined for any $b > 0$.

Then by Theorem 1, we get that the linear control system **(5)** is controllable on [0,b].

Let $F\big(t, z(t)\big) = \int_0^t h(t, s, z(s)) ds$, then obviously that Q satisfies the hypothesis (I).

Therefore, by Theorem 3 the system (14) is controllable on [0,b].

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