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Solution of Heat Equation by Fourier Transform and Wavelet Transform Comparison Study

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ABSTRACT

In this paper, we restrict ourselves to compare the solution of the one-dimensional heat equation which is produced through implementation of both Fourier and Wavelet transforms.

INTRODUCTION

Many methods have been developed so far for solving the heat equation. The well _known numerical methods for solving PDEs are: weighted residual techniques, the finite difference methods, the finite element methods and the boundary element methods.

The one- dimensional heat equation with Dirichlet boundary conditions is defined as (A.Tveito and R.Winther, 1991)(A.M. Wazwaz, 2009):

$$u_t = \alpha u_{xx} + q(x,t) 0 \le x \le L, \quad t \ge 0 \tag{1}$$

With initial and boundary conditions:

$$u(x,0) = f(x) \tag{2}$$

$$u(0,t) = h_0(t), \; u(L,t) = h_1(t) \tag{3}$$

Where $u = u(x\,,t)$ represents the temperature at the position x and time t, α is the *thermaldiffusivity* of the material,that depends on specific heat of the material and the cross section of the material rod.

Heat equation mainly in one-dimension had been studied by many authors (Dabral V, Kapoor S andDhawan S, 2011). A comparative study between the traditional separation of variables method and Adomian method for heat equation had been examined by Gorguis and Chan (Gorguis A and Chan WKB, 2008). Dehghan(M. Dehghan, 2000) considered the use of second-order finite difference scheme to solve the two-dimensional heat equation. After that, Mohebbi and Dehghan (Mohebbi A andDehghan M, 2010) presented a fourth-order compact finite difference approximation and cubic C1-spline collocation method for the solution with fourth-order accuracy in both space and time variables. Recently Dabral et al. (Dabral V, Kapoor S andDhawan S, 2011), propose B-spline finite element method to get numerical solutions of one dimensional heat Equation.

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Wavelet theory (Lepik U, 2011) (S.G. Venkatesh, S.K. Ayyaswamy, S.R. Balachandar, K. Kannan, 2012) is one of the relatively new techniques which is being utilized for solving wide range of physical problems related to various branches of engineering and applied sciences. Wavelet methods have been applied for solving PDEs from beginning of the early 1990s (Lepik U, 2011). With the passage of time, lot of rapid developments is taking place which are helpful in increasing the accuracy of this scheme. The most common related schemes are Haar Wavelets (K. Maleknejad and F. Mirzaee, 2005), Harmonic Wavelets of successive approximation (C. Cattani and A. Kudreyko, 2010), CAS Wavelets (S.G. Venkatesh, S.K.Ayyaswamy, S.R. Balachandar, 2013), Legendre Wavelets (F. Mohammadi and M.M. Hosseini, 2010) and Chebyshev Wavelets (E. Babolian and F. Fattah Zadeh, 2007). In the similar context, we merge Chebyshev polynomials with the traditional wavelet technique (L. Nanshan and L.E. Bing, 2009)(S.G. Venkatesh, S.K. Ayyaswamy, S.R. Balachandar, K. Kannan, 2012) used for finding solutions of partial differential equations. The modified version which is called Chebyshev Wavelets transform proves to be fully compatible with the complexity of the given problems and obtained results are extremely accurate. In particular, we apply Chebyshev Wavelets transformon homogeneous heat equations. It is observed that is very user friendly but is extremely accuratecomparison with Fourier transform. The error estimates explicitly reveal the very high accuracy level of the suggested technique

2. *Solvingthe one- dimensional heat equation with finite Fourier transform:*

For solvingheateqs. (1-3)on finite interval$0 < x < L$, by means of finite Fourier transform. In particular, we use finite sine Fourier transform with Dirichlet boundary conditions, which is definedby (J.Lambers, 2013-2014)

$$\widehat{u_s}(x,t) = \int_0^L u(x,t)\sin\left(\frac{n\pi x}{L}\right)dx \quad ,n = 1,2,..., \tag{4}$$

with its inverse finite sine Fourier transform

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\widehat{u_s}(x,t)\,sin\left(\frac{n\pi x}{L}\right) \tag{5}$$

Applying the properties of the Fourier transform derivatives (J.Lambers, 2013-2014),the initial and boundary conditions (2) and (3)to get the solution ofequation (1) as:

$$u(x,t) = v(x,t) + \sum_{n=1}^{\infty} I_1\,sin\left(\frac{n\pi x}{L}\right)e^{-\alpha(n\pi/L)^2 t}$$
$$+\sum_{n=1}^{\infty} sin\left(\frac{n\pi x}{L}\right)\int_0^t I_1\,e^{-\alpha(n\pi/L)^2(t-\tau)}d\tau \tag{6}$$

where $v(x,t), I_1$ and I_2aregiven by

$$v(x,t) = h_0(t) + \frac{x}{L}[h_1(t) - h_0(t)] \tag{7}$$

$$I_1 = \frac{2}{L}\int_0^L\big(f(x) - v(x,0)\big)\sin\left(\frac{n\pi x}{L}\right)dx \tag{8}$$

$$I_2 = \frac{2}{L}\int_0^L(q(x,\tau) - v_t + \alpha v_{xx})\sin\left(\frac{n\pi x}{L}\right)dx, 0 \le \tau \le 1 \tag{9}$$

3. Wavelets and theChebyshev Wavelets:

Wavelets constitute a family of functions constructed from dilation and translation of a single $\psi(t)$ called the mother wavelet. When the dilation parameter *a* and the translation parameter *b* vary continuously we have the following family of continuous Wavelets (Morlet, J. ,Arens, G.,Fourgeau, E., and Giad,D, 1982):

$$\psi_{a,b}(t) = |a|^{-1\backslash 2}\psi\left(\frac{t-b}{a}\right), a,b \in R, a \neq 0 \tag{10}$$

Restrict the parameters *a* and *b* to discrete values

as $a = a_0^{-k}, b = nb_0a_0^{-k}, \quad a_0 > 1, \quad b_0 > 0$` ,

to get the following family of discrete wavelets

$$\psi_{k,n}(t) = |a_0|^{k/2}\psi(a_0^k t - nb_0), \quad k,n \in \mathbb{Z} \tag{11}$$

Where $\psi_{k,n}$form a wavelet basis for $L^2(\mathbb{R})$.

In particular, we use the second Chebyshev $\psi_{k,n}(t)$ waveletwhich is orthonormal basis when $a_0 = 2$ and $b_0 = 1$.

The second Chebyshev wavelets $\psi_{k,n}(t)$=$\psi(k,n,m,t)$ involving four arguments and$n = 1,\cdots,2^{k-1}$, *k* is any positive integer, *m* is the degree of the secondChebyshev polynomials and*t*is the normalized time. They are defined on the interval [0, 1] as follows:

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\widetilde{T}_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}} \\ 0 & otherwise \end{cases} \tag{12}$$

where

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$$\tilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}} & m = 0 \\ \sqrt{\frac{2}{\pi}} T_m(t) & m \geq 1 \end{cases} \quad (13)$$

m = 0,1,…,M -1; and *M* is a fixed positive integer. Here$T_m(t)$ are the second Chebyshev polynomials of degree m with respect to the weight function $w(t) = 1/\sqrt{1-t^2}$ on the interval [-1, 1], and satisfy the following recursive formula:

$T_0(t) = 1;\ T_1(t) = t$

$$T_{m+1}(t) = 2tT_m - T_{m-1}\ , m = 1,2,3, ... \quad (14)$$

Note that in dealing with the Chebyshev polynomials wavelets, the weight function $\tilde{w}(t) = w(2t-1)$.The Chebyshev wavelets are an orthogonal set with respect to the weight functions $w_n = w(2^k t - 2n + 1)$.

A function*f*(*t*) defined over*[0, 1]* may be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{nm} \psi_{nm}(t) \quad (15)$$

Where

$$C_{nm} = \langle f(t), \psi_{nm}(t) \rangle = \int_o^1 w_n \psi_{nm}(t) f(t) dt \quad (16)$$

In which $\langle .,. \rangle$ denotes the inner product in $L^2[0,1]$.

If the infinite series in eq. (15) is truncated, then it can be written as

$$f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(t) = C^T \Psi(t), \quad (17)$$

Where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$C = \left[c_{10}, c_{11}, \cdots, c_{1(M-1)}, c_{20}, \cdots, c_{2(M-1)}, \cdots, c_{2^{k-1}0}, \cdots, c_{2^{k-1}(M-1)}\right]^T \quad (18)$$

And

$$\Psi(t) = \left[\psi_{10}, \psi_{11}, \cdots \psi_{1(M-1)}, \psi_{20}, \cdots, \psi_{2(M-1)}, \cdots, \psi_{2^{k-1}0}, \cdots, \psi_{2^{k-1}(M-1)}\right]^T \quad (19)$$

So we can write in the form

$$f(t) \simeq \sum_{i=1}^{m} \sum_{i=1}^{m} c_i \psi_i(t) = C^T \Psi(t), i = 1, \dots, m \quad (20)$$

Where $c_i = c_{nm},\ \ \psi_i(t) = \psi_{nm}(t),\ \ m = 2^{K-1}M,\ \ i = M(n-1) + m + 1$ (21)

Similarly, an arbitrary function of two variables $u(x,t) \in L^2(\mathbb{R} \times \mathbb{R})$ defined over$[0,1] \times [0,1]$, may be expanded to two Chebyshev wavelets basis as,

$$u(x,t) \approx \sum_{i=1}^{m} \sum_{j=0}^{m} a_{ij} \psi_i(x) \psi_j(t) = \Psi^T(x) A \Psi(t) \quad (22)$$

Where$A = [a_{ij}]$*and* $a_{ij} = \langle \psi_i(x), \langle u(x,t), \psi_j(t) \rangle \rangle$, (23)

To obtain the discrete form put the collocation points

$$x_i = \frac{2i-1}{2m}, \qquad i = 1,2, \dots\dots, m. \quad (24)$$

The Chebyshev wavelets operational matrix is given by:

$$\Phi^T(x)_{m \times m} = \left[\Psi\left(\frac{1}{2m}\right), \Psi\left(\frac{3}{2\boldsymbol{m}}\right), \dots.., \Psi(\frac{2m-1}{2m})\right]^T \quad (25)$$

4. The operational matrix of integration of Chebyshev wavelets:

In this section we present the integration operation matrix of the Chebyshev Wavelets [4] by integrating the vector $\Psi(t)$ which is defined in eq.(19), to get

$$\int_0^t \Psi(s) ds \simeq P \Psi(t) \quad (26)$$

In general $P$is a $2^{k-1}M \times 2^{K-1}M$ matrix given by

$$P = \begin{bmatrix} C & S & S & \dots & S \\ 0 & C & S & \dots & S \\ 0 & 0 & C & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & S \\ 0 & 0 & 0 & \cdots & C \end{bmatrix} \quad (27)$$

Where S and C are $M \times M$ matrices given by:

$$S=\frac{\sqrt{2}}{2^K}\begin{bmatrix} 1 & 0 & 0 & & 0\\ 0 & 0 & 0 & & 0\\ -\frac{1}{3} & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & & 0\\ -\frac{1}{15} & 0 & 0 & & 0\\ \vdots & & & \ddots & \vdots\\ -\frac{1}{M(M-2)} & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{28}$$

And

$$C=\frac{1}{2^k}\begin{bmatrix} \frac{1}{2}\frac{1}{2\sqrt{2}} & & 0 & 0 & & 0 & 0 & 0\\ -\frac{1}{8\sqrt{2}} & 0 & \frac{1}{8} & 0 & \cdots & 0 & 0 & 0\\ -\frac{1}{6\sqrt{2}} & -\frac{1}{4} & 0 & \frac{1}{12} & & 0 & 0 & 0\\ & & & & \ddots & & & \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots\\ -\frac{1}{2\sqrt{2}(M-1)(M-3)} & 0 & 0 & 0 & \cdots & -\frac{1}{4(M-3)} & 0 & -\frac{1}{4(M-1)}\\ -\frac{1}{2\sqrt{2}(M-2)} & 0 & 0 & 0 & & 0 & -\frac{1}{4(M-2)} & 0 \end{bmatrix} \tag{29}$$

5. Operational matrix of Derivative for Chebyshev Wavelet:

In this section a review operational matrix of derivative for Chebyshev wavelets will be made (F.Mohammadi, 2014).

The derivative of this vector $\Psi\ \ (x)$ can be expressed by

$$\frac{d\Psi}{dx}=D\Psi\ \ (x) \tag{30}$$

where D is the 2^kM operational matrix of the derivative defined as follow

$$D\ =\begin{bmatrix} F & 0 & \cdots & 0\\ 0 & F & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & F \end{bmatrix} \tag{31}$$

In which F is $M\times M$ matrix and its entries (r, s) the element is defined as follow

$$F_{r,s}=\begin{cases} 2^{k+2}s\sqrt{\frac{\gamma_{r-1}}{\gamma_{s-1}}}\, r=2,\dots,M, s=1,\dots,r-1 and\ \ (r+s) odd\\ 0 \qquad otherwise \end{cases} \tag{32}$$

Where

$$\gamma_m=\begin{cases} 2 & m=0\\ 1 & m\geq 1 \end{cases} \tag{33}$$

6. Solving one- dimensional heat equation with Chebyshev Wavelet:

For solving the heat eq. (1-3) on a finite interval $0<x<L$, by Wavelet transform. In particular, we use Chebyshev Wavelets transform, which is defined in equations (18)and(19), we first approximate the initial condition *f(x)* by *m* terms of the Chebyshev Wavelets as follows:

$f(x)\simeq \Psi^T(t)F\Psi(x)$, where F is known $m\times m$ matrix given by:

$$F=\begin{pmatrix} f_0 & f_1 & . & . & . & f_{m-1} & f_m\\ 0 & 0 & . & . & . & 0 & 0\\ . & . & & & & . & .\\ . & . & & & & . & .\\ . & . & . & . & . & . & .\\ 0 & 0 & . & . & . & 0 & 0 \end{pmatrix} \tag{34}$$

Also, we approximate the solution *u(x,t)* and *q(x,t)* by the Chebyshev waveletsas:

$u(x,t)=\Psi^T(t)A\Psi(x)$(35)

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$q(x,t) = \Psi^T(t)Q\Psi(x)$ (36)

Where Q is known $m \times m$ matrix, but A is an $m \times m$ unknown matrix. Now integrating eq.(1) from 0 to t and using the initial condition (2) ,as:

$u(x,t) - f(x) = \int_0^t u_{xx}\,dt + \int_0^t q(x,t)\,dt$ (37)

Also using eqs.(26),(30),(35)and(36) we obtain

$\Psi^T(t)A\Psi(x) - \Psi^T(t)F\Psi(x) = \Psi^T(t)P^T AD^2\Psi(x) + \Psi^T(t)P^T Q\Psi(x)$ (38)

The residual $R(x,t)$ for eq. (38) can be written as:

$R(x,t) = \Psi^T(t)(A - F - P^T AD^2 - P^T Q)\Psi(x)$ (39)

$R(x,t) = \Psi^T(t)E\Psi(x)$(40) Where $E = A - F - P^T AD^2 - P^T Q$ (41)

As in a typical tau method (Canuto C., Hussaini M.Y., Quarteroni A. and Zang T.A., 1988) an $m\times m$ linear algebraic equations may be generated using the following algebraic equations

$E_{ij} = 0$ *i=1,2,…,m, j=1,2,…,m-2* (42)

Also, substituting eq.(35) intothe boundary conditions eqs.(3) will give:

$\Psi^T(t)A\Psi(0) = h_1(t)$ (43)

$\Psi^T(t)A\Psi(1) = h_2(t)$ (44)

Equations (43) and (44) are collocated at Chebyshev wavelets t_i, i=1,2,…,m of $\Psi(t)$. The unknown coefficients a_{ij}, i=1,…,m, j=1,…,n can be evaluated from eqs.(42)-(44). Consequently, the $u(x,t)$ given in eq.(35) can be calculated.

In this section, numerical example of the heat equation in form (1) with the initial and boundary conditions (2-3) with the Chebyshev Wavelets method are investigated. To show the efficiency of the present method, we report the root mean square error for Fourier method and Chebyshev Wavelets method.All programs have been performed by Matlab (v.7.11.0)

Example:

In this example, we consider the classical heat equation in (1) with

$\alpha = 1$and $q(x,t) = 0$.The initial and boundary conditions are givenby

$u(x,0) = \sin(\pi x)$ $0 < t < 1$

$u(0,t) = 0,\quad u(1,t) = 0$$0 \le x \le 1$

The exact solution of this problem is $u(x,t) = \sin(\pi x)e^{-\pi^2}$. The root-mean-square errorfor some different values of *m=6,10,8,12,16,20*are presented in Table 1 and Figure 1.

Table 1:

m	*MSE Fourier*	*MSE Wavelet*
6	0.2967	0.008091
8	0.3273	0.005261
10	0.3462	0.001282
12	0.3591	0.001105
16	0.3756	0.011034
20	0.3857	0 0.000018

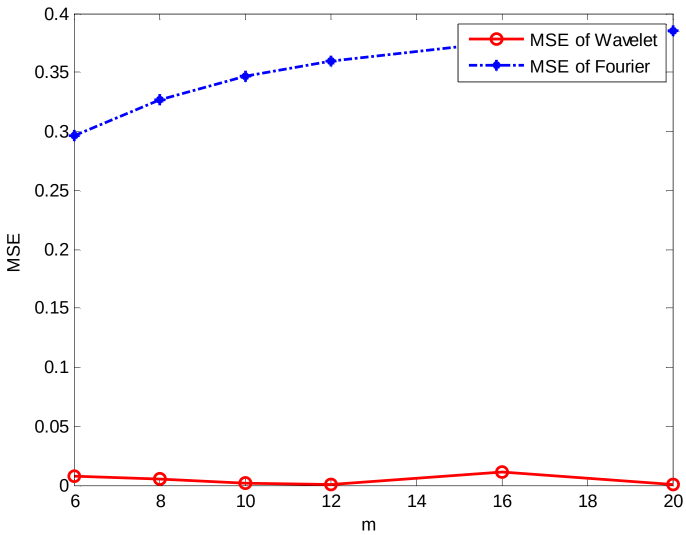

Fig. 1:

Conclusion:

It has been well shown that in applying the properties of wavelet transform the differential equation can be solved conveniently and accurately. The key idea is to transform the partial differential equation into a group of algebraic equations which involves finite number of variables.In wavelettransform we get more accurate local description of the solution. The wavelet coefficientmatrix symbolizes the components that are their selves local and are easier to interpret. In Fourier transform we lose the information of the localization of the solution.

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