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The direct theorem for multiplier approximation by two dimensional multiplier Bernstein Durrmeyer operators in L_{p,λ_n} space .

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ABSTRACT

In this paper, we established the degree of multiplier approximation of unbounded function in to two dimensional space by multiplier Bernstein- Durrmeyer operators in terms of K-functional in L_{p,λ_n} space .

INTRODUCTION

In 1967, Durrmeyer introduced Bernstein- Durrmeyer operators, for $f\in L_1[0,1]$ there are many papers which studied their properties (Feilong, C., 2005; DING-XUAN ZHOU, 1994; Ding-Xuan Zhou, 1990; Ding-Xuan Zhou, 1992; Berdysheva, E.E. and Bing-Zheng Li, 2014), M. Heilmann (1988) studied the saturation of that operator in L_p space.in 1991, Zhang studied the characterization of convergence for $M_{n,1}(f,x)$with Jacobi weights Z. Ditzian and K. G. Ivanov (1991) and (Felten, M.,) gave the inverse results in L_p and

C[0, 1], H. Berens (1992) gave the inverse theorem for a modified form of these operators with Jacobi weights.

The main objective for our paper is to study the best multiplier approximation of unbounded functions byBernstein- Durrmeyer operator, we will introduce the direct theorem for the multiplier approximation of this function by $(\mathcal{M}_n f\lambda_n(\varkappa,y))$the multiplier Bernstein- Durrmeyer operator in terms of multiplier K-functional.

The main results of this paper is to approximation error of unbounded functions in multiplier integral norm such that we prove a direct estimate.

The one dimensional Bernstein- Durrmeyer operator are defined as:

$$\mathcal{M}_n f(x) = (n+1)\sum_{k=0}^{n} p_{nk}(x)\int_0^1 p_{nk}(s)f(s)\,ds \qquad (1,1)$$

Where$f\in L_p[0,1]\,, p\geq 1$ and

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$p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ (1, 2)

For $\varkappa = (\varkappa_1, \varkappa_2) \in R^2$, $k = (k_1, k_2) \in N_0^{;2}$ $and\ n \in N$

We will write $|\varkappa| = \varkappa_1 + \varkappa_2$,

$\varkappa^k = \varkappa_1^{k_1} \varkappa_2^{k_2}, |k| = k_1 + k_2, k! = k_1!\, k_2!$

And $\binom{n}{k} = \binom{n}{k_1 k_2} = \frac{n!}{k!(n-|k|)!}$,

$\sum_{i=0}^{k} = \sum_{i_1}^{k_1} \sum_{i_1}^{k_2}$

T.Walter,1973. Define the multiplier convergence as follows

A series $\sum_{n=0}^{\infty} a_n L_n(x)$ is called a multiplier convergence if there is a sequence $\{\lambda_n(\mathrm{x})\}$ such that $\sum_{n=0}^{\infty} a_n L_n(x) \lambda_n(x) < \infty$, λ_n is called a multiplier for the convergence.

From the definition of multiplier convergence we can define the multiplier for the integral norm

For any function *f*, we can define the multiplier integral norm as follows:

$$\|f\|_{p,\lambda_n} = sup\left\{\left(\int_{\mathfrak{H}} |f \lambda_n(\varkappa)|^p dx\right)^{1/p} : \varkappa \in \mathfrak{H}, \mathrm{p} \geq 1\right\} \quad (1, 3)$$

$\lambda_n$is called the the multiplier for the integral norm.

Let us define the space of all these functions as:

$$L_{p,\lambda_n} = \left\{f : f : (0, \infty] \to \mathcal{R}\ such\ that\ \int\ f \lambda_n(x) dx < \infty\right\} \quad (1, 4)$$

The degree of best multiplier approximation to a given continuous function with respect to trigonometric or algebraic polynomials is given by:

$$\mathrm{E_n}(f)_{\mathrm{c},}\lambda_\mathrm{n} = \inf\{\|f - \mathrm{p}\|_{\mathrm{c},\lambda_\mathrm{n}} : \mathcal{P} \in \mathcal{G}_\mathrm{n}\} \quad (1, 5)$$

The degree of best multiplier approximation to a given multiplier integral function with respect to trigonometric or algebraic polynomials is given by:

$$\mathrm{E_n}(f)_{\mathrm{p},}\lambda_\mathrm{n} = \inf\{\|f - \mathrm{p}\|_{\mathrm{c},\lambda_\mathrm{n}} : \mathcal{P} \in \mathcal{G}_\mathrm{n}\} \quad (1, 6)$$

Where $\mathcal{G}_\mathrm{n}$ the set of all trigonometric (algebraic) polynomials.

We shall make some notations,

Let $W_{p,\lambda_n} = \left\{f\ \in L_{p,\lambda_n}(\mathfrak{H}) : \frac{\partial}{\partial x} f \lambda_n, \frac{\partial}{\partial y} f \lambda_n, \Omega_i(f)_{p,\lambda_n} < \infty\ \right\}$ (1,7)

For$f \in L_{p,\lambda_n}$,

$$K(f,t)_{p,\lambda_n} = \inf\{\|f - g\|_{p,\lambda_n} + t\Omega(g)_{p,\lambda_n}\} \quad (1,8)$$

Is the so called K- functional.

Now let

$\mathfrak{H} = \{\varkappa = (\varkappa_1, \varkappa_2) : \varkappa_1 + \varkappa_2 \leq 1, \varkappa_1, \varkappa_2 \geq 0\}$be the two- dimensional simplex.

The transformation T: $\mathfrak{H} \to \mathfrak{H}$ defined by

$$\mathrm{T}(\varkappa_1, \varkappa_2) = (t_1, t_2),\ t_i = \begin{cases} \varkappa_j & i = j \\ 1 - |\varkappa| & i \neq j \end{cases} \quad (19)$$

$$\mathcal{M}_n f(\varkappa) = \mathcal{M}_n(f_T; T_\varkappa) = \mathcal{M}_n(f_T; \varkappa) \quad (1,10)$$

$$T_1(\varkappa_1, \varkappa_2) = (1 - \varkappa_1 - \varkappa_2,\ \varkappa_2),$$
$$T_2(\varkappa_1, \varkappa_2) = (\varkappa_1, 1 - \varkappa_1 - \varkappa_2) \quad (1,11)$$

$$f_1(\varkappa_1, \varkappa_2) = f(1 - \varkappa_1 - \varkappa_2,\ \varkappa_2),$$
$$f_2(\varkappa_1, \varkappa_2) = f(\varkappa_1, 1 - \varkappa_1 - \varkappa_2) \quad (1,12)$$

And the norm $\Omega(f)_{p,\lambda_n} = sup_{i=0,1,2} \Omega_i(f)_{p,\lambda_n} + \|f\|_{p,\lambda_n}$ where

$\Omega_i(f)_{p,\lambda_n} = \Omega_0(f_i)_{p,\lambda_n}$for i=1, 2 (1, 13)

And $\Omega_0(f_i)_{p,\lambda_n} =$

$$sup_{\varkappa_1+\varkappa_2 \leq \frac{3}{4}} \left\{ \begin{array}{c} \left\| \varkappa_1 \frac{\partial^2}{\partial \varkappa_1{}^2} f \lambda_n(\varkappa_1, \varkappa_2) \right\|_{L_p\left(\varkappa_1+\varkappa_2 \leq \frac{3}{4}\right)}, \\ \left\| \varkappa_2 \frac{\partial^2}{\partial \varkappa_2{}^2} f \lambda_n(\varkappa_1, \varkappa_2) \right\|_{L_p\left(\varkappa_1+\varkappa_2 \leq \frac{3}{4}\right)}, \\ \left\| (\varkappa_1 \varkappa_2)^{1/2} \frac{\partial^2}{\partial \varkappa_1 \partial \varkappa_2} f \lambda_n(\varkappa_1, \varkappa_2) \right\|_{L_p\left(\varkappa_1+\varkappa_2 \leq \frac{3}{4}\right)} \end{array} \right\} \quad (1, 8)$$

We observe that $\Omega_0(f_1)_{p,\lambda_n}$=$\Omega_1(f)_{p,\lambda_n}$, for example

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$$\Omega_1(f)_{p,\lambda_n} = \Omega_0(f_1)_{p,\lambda_n} = sup_{\varkappa\geq\frac{1}{4}} \left\{ \begin{array}{c} \left\| 1-\varkappa_1-\varkappa_2 \frac{\partial^2}{\partial {\varkappa_1}^2} f\lambda_n(\varkappa_1,\varkappa_2)\right\|_{L_p}, \\ \left\| \varkappa_2 \frac{\partial^2}{\partial {\varkappa_2}^2} f\lambda_n(\varkappa_1,\varkappa_2)\right\|_{L_p}, \\ \left\| \frac{1}{(\varkappa_2(1-\varkappa_1-\varkappa_2))^{1/2}} \frac{\partial^2}{\partial\varkappa_1\partial\varkappa_2} f\lambda_n(\varkappa_1,\varkappa_2)\right\|_{L_p} \end{array} \right\} \quad (1, 14)$$

We fix some constants $\frac{2}{3} < a = \frac{33}{48} < b = \frac{17}{24} < c < \frac{3}{4}$

For $f \in L_{p,\lambda_n}(\mathfrak{H})$, the two dimensional Bernstein- Durrmeyeroperators are given by:

$$\mathcal{M}_n f\lambda_n(\varkappa_1,\varkappa_2) = (n+1)(n+2)\sum_{k+m\leq n} \wp_{n,k,m}(\varkappa_1,\varkappa_2) \iint \wp_{n,k,m}(s,t) f\lambda_n(s,t)\, ds\, dt \quad (1,15)$$

Where

$$\wp_{n,k,m}(\varkappa_1,\varkappa_2) = \binom{n}{k}\binom{n-k}{m} {\varkappa_1}^k {\varkappa_2}^m (1-\varkappa_1-\varkappa_2)^{n-k-m} \quad (1,16)$$

The modified Bernstein-Durrmeyer operators (DING-XUAN ZHOU, 1994)

$$\mathcal{M}_n^* h(\varkappa) = \sum_{k\leq n} \wp_{n,k}(\varkappa)(n+2)\int_0^1 \wp_{n+1,k}(t) f(t) dt \quad (1, 17)$$

Auxiliary result:

We need the following lemmas to prove the main result

Lemma(2.1):

For $\mathcal{M}_n f\lambda_n(\varkappa_1,\varkappa_2)$ given in (1, 10) we have

$$\frac{\partial}{\partial\varkappa_1}\mathcal{M}_n f\lambda_n(\varkappa_1,\varkappa_2) = (n+1)(n+2)\sum_{k=0}^{n}\sum_{k=m}^{n-k} \frac{n!}{k!m!(n-k-m)!} {\varkappa_2}^m (-{\varkappa_1}^k (n-k-m)(1-\varkappa_1-\varkappa_2)^{n-k-m-1} + (1-\varkappa_1-\varkappa_2)^{n-k-m} k{\varkappa_1}^{k-1}) \int_{\mathfrak{H}}\int_{\mathfrak{H}} \wp_{n,k,m}(s,t) f\lambda_n(s,t)\, dsdt$$

$$= (n+1)(n+2)\sum_{k=0}^{n}\sum_{k=m}^{n-k} \frac{n(n-1)!}{k(k-1)!m!(n-k-m)!} k \int_{\mathfrak{H}}\int_{\mathfrak{H}} \wp_{n,k-1,m}(s,t) f\lambda_n(s,t)\, dsdt \; {\varkappa_2}^m {\varkappa_1}^{k-1}(1-\varkappa_1-\varkappa_2) - (n-k-m){\varkappa_1}^k \quad (2, 1)$$

$$= \mathrm{n}(n+1)(n+2)\sum_{k=0}^{n}\sum_{k=m}^{n-k} \wp_{n-1,k-1,m}(\varkappa_1,\varkappa_2) \int_{\mathfrak{H}}\int_{\mathfrak{H}} \wp_{n-1,k-1,m}(s,t) f\lambda_n(s,t)\, dsdt - \mathrm{n}(n+1)(n+2)\sum_{k=0}^{n}\sum_{k=m}^{n-k} \wp_{n-1,k-1,m}(\varkappa_1,\varkappa_2) \int_{\mathfrak{H}}\int_{\mathfrak{H}} \wp_{n-1,k-1,m}(s,t) f\lambda_n(s,t)\, dsdt - \big(n(n+1)(n+2)\big)$$

$$\sum_{k=0}^{n}\sum_{k=m}^{n-k} \wp_{n-1,k-1,m}(\varkappa_1,\varkappa_2) \int_{\mathfrak{H}}\int_{\mathfrak{H}} \wp_{n,k-1,m}(s,t) f\lambda_n(s,t)\, dsdt$$

$$\sum_{k=0}^{n}\sum_{k=m}^{n-k} n\wp_{n-1,k-1,m}(\varkappa_1,\varkappa_2)\Big(\int_{\mathfrak{H}}\int_{\mathfrak{H}} (n+1)(n+2)\, \wp_{n,k,m}(s,t) f\lambda_n(s,t)\, dsdt - \int_{\mathfrak{H}}\int_{\mathfrak{H}} \wp_{n,k-1,m}(s,t) f\lambda_n(s,t)\, dsdt$$

$$\frac{\partial^2}{\partial^2\varkappa_2}\mathcal{M}_n f\lambda_n(\varkappa_1,\varkappa_2) = n(n-1)\sum_{k=0}^{n}\sum_{k=m}^{n-k} \frac{n!}{k!m!(n-k-m)!} {\varkappa_2}^m {\varkappa_1}^{k-2}(1-\varkappa_1-\varkappa_2)^{n-k-m-1}$$

$\int_{\mathfrak{H}} \int_{\mathfrak{H}} \wp_{n,k-1,m}(s,t) f\lambda_n(s,t)\, dsdt$
$\left(k(k-1)(1-\varkappa_1-\varkappa_2)^2 - 2k(n-k-m)\varkappa_1(1-\varkappa_1-\varkappa_2) + (n-k-m)\right)$
$(n-k-m-1){\varkappa_1}^2$ (2,2)
$= n(n-1)$
$\sum_{k=0}^{n}\sum_{k=m}^{n-k} \wp_{n-2,k-2,m}(\varkappa_1,\varkappa_2)\, F_{n,k,m} - 2F_{n,k-1,m} + F_{n,k-2,m}$

Where $\int_{\mathfrak{H}} \int_{\mathfrak{H}} \wp_{n,k,m}(s,t) f\lambda_n(s,t)\, dsdt = F_{n,k,m}$

$\frac{\partial^2}{\partial\varkappa_1\, \partial\varkappa_2}\mathcal{M}_n f\lambda_n(\varkappa_1,\varkappa_2) = (n+1)(n+2)\sum_{k=0}^{n}\sum_{k=m}^{n-k}\frac{n!}{k!m!(n-k-m)!}$
$\int_{\mathfrak{H}} \int_{\mathfrak{H}} \wp_{n,k,m}(s,t) f\lambda_n(s,t)\, dsdt\, {\varkappa_2}^{m-1}{\varkappa_1}^{k-1}(1-\varkappa_1-\varkappa_2)^{n-k-m-2}$
$(km(1-\varkappa_1-\varkappa_2)^2 - (k\varkappa_2+m\varkappa_1)(n-k-m)(1-\varkappa_1-\varkappa_2)$
$+(n-k-m)(n-k-m+1)\varkappa_1\varkappa_2)$ (2, 3)
$= n(n-1)\sum_{k=1}^{n}\sum_{m=1}^{n-k}\wp_{n-2,k-1,m-1}(\varkappa_1,\varkappa_2)\left(F_{n,k,m} - F_{n,k-1,m} - F_{n,k,m-1} + F_{n,k-1,m-1}\right).$

Lemma(2.2):

Let $h \in L_{p,\lambda_n}([0,1])$ *and* $h^{`}(\varkappa), h^{``}(\varkappa)$ *exists, we have*

$\|\mathcal{M}_n^* h(\varkappa) - h(\varkappa)\|_{p\lambda_n} \le C_p\left(\|h(\varkappa)\|_{p\lambda_n} + \|\varkappa(1-\varkappa)h^{``}(\varkappa)\|_{p\lambda_n}\right)/(n-k+1)^p$ ...*(2,4)*

Where C_p is a constant independent of n, k and h

Proof:

Using Tylor`s expansion at t= $\varkappa$

$$h\lambda_n(t) = h\lambda_n(\varkappa) + (t-\varkappa)\left(h\lambda_n(\varkappa)\right)^{`} + {}^1\!/_2\,(t-\varkappa)^2\left(h\lambda_n(\varkappa)\right)^{``}(\theta)$$

$t \le \theta \le \varkappa$, we get

$$\mathcal{M}_n^* h\lambda_n(\varkappa) - h\lambda_n(\varkappa) = \mathcal{M}_n^*(t-\varkappa,\varkappa)\left(h\lambda_n(\varkappa)\right)^{`} + \frac{1}{2}\mathcal{M}_n^*((t-\varkappa)^2,\varkappa)\left(h\lambda_n(\varkappa)\right)^{``}(\theta)$$

Taking the multiplier norm for both sides to get

$$\|\mathcal{M}_n^* h(\varkappa) - h(\varkappa)\|_{p\lambda_n}$$
$$\le sup\left(\int_0^1 \left|\mathcal{M}_n^*(t-\varkappa,\varkappa)\left(h\lambda_n(\varkappa)\right)^{`}\right|^p d\varkappa\right)^{\frac{1}{p}}$$
$$+\frac{1}{2}sup\left(\int_0^1 \left|\mathcal{M}_n^*((t-\varkappa)^2,\varkappa)\left(h\lambda_n(\varkappa)\right)^{``}(\theta)\right|^p d\varkappa\right)^{\frac{1}{p}}$$

Using Holder`s inequality, we get

$$\|\mathcal{M}_n^* h(\varkappa) - h(\varkappa)\|_{p\lambda_n} \le \mathcal{M}_n^*(t-\varkappa,\varkappa) sup\left(\int_0^1 \left|\left(h\lambda_n(\varkappa)\right)^{`}\right|^p d\varkappa\right)^{\frac{1}{p}}$$
$$+\frac{1}{2}\mathcal{M}_n^*((t-\varkappa)^2,\varkappa) sup\left(\int_0^1 \left|\left(h\lambda_n(\varkappa)\right)^{``}(\theta)\right|^p d\varkappa\right)^{\frac{1}{p}}$$

By (Berdysheva, E.E. and Bing-Zheng Li, 2014), let m∈ $N_{0,}$ the moment of the operator $\mathcal{M}_n f(\varkappa)$ of order m is defined by

$$T_{n,m}(\varkappa) = (n+1)\sum_{k=\mathfrak{H}}^{n}\wp_{nk}(\varkappa)\int_0^1 \wp_{nk}(t)(t-\varkappa)^m\, dt \quad (2, 5)$$

Then

$$\|\mathcal{M}_n^* h(\varkappa) - h(\varkappa)\|_{p\lambda_n}$$
$$\le (n+1)\sum_{k=\mathfrak{H}}^{n}\wp_{nk}(\varkappa)\int_0^1 \wp_{nk}(t)(t-\varkappa)\, dt sup\left(\int_0^1 \left|\left(h\lambda_n(\varkappa)\right)^{`}\right|^p d\varkappa\right)^{\frac{1}{p}}$$
$$+\frac{1}{2}(n+1)\sum_{k=\mathfrak{H}}^{n}\wp_{nk}(\varkappa)\int_0^1 \wp_{nk}(t)(t-\varkappa)^2\, sup\left(\int_0^1 \left|\left(h\lambda_n(\varkappa)\right)^{``}(\theta)\right|^p d\varkappa\right)^{\frac{1}{p}}$$

$$\leq T_{n,1}(\varkappa) sup\left(\int_0^1 \left|\left(h\lambda_n(\varkappa)\right)^{`}\right|^p d\varkappa\right)^{\frac{1}{p}} + \frac{1}{2}T_{n,2}(\varkappa) sup\left(\int_0^1 \left|\left(h\lambda_n(\varkappa)\right)^{``}(\theta)\right|^p d\varkappa\right)^{\frac{1}{p}}$$

Since $\frac{1}{2}T_{n,2}(\varkappa) \leq T_{n,1}(\varkappa)$, $\forall n \geq 3$[9].

We get:

$$\leq A\left(sup\left(\int_0^1 \left|\left(h\lambda_n(\varkappa)\right)^{`}\right|^p d\varkappa\right)^{\frac{1}{p}} + sup\left(\int_0^1 \left|\left(h\lambda_n(\varkappa)\right)^{``}(\theta)\right|^p d\varkappa\right)^{\frac{1}{p}}\right)$$

$$\leq A\left[\left\|h^{`}\right\|_{L_{p\lambda_n}} + \left\|h^{``}\right\|_{L_{p\lambda_n}}\right]$$

$$\leq A\left\|h^{`}\right\|_{L_{p\lambda_n}} \leq A\left\|h^{``}\right\|_{L_{p\lambda_n}}$$

Lemma(2.3):

For p≥ 1 and$f \in L_{p\lambda_n}(\mathfrak{H})$, we have

$$\Omega\left(\mathcal{M}_n f(\varkappa_1, \varkappa_2)\right)_{L_{p,\lambda_n}} \leq (80)^p (n)^p \|f\|_{p,\lambda_n} \quad (2, 6)$$

Proof:

It is enough to estimate $\Omega_1\left(\mathcal{M}_n f(\varkappa_1, \varkappa_2)\right)_{L_{p,\lambda_n}}$

Let $q = \frac{p}{1-p}$, then

$$\left\|\varkappa_1 \frac{\partial^2}{\partial^2 \varkappa_2} \mathcal{M}_n f\lambda_n(\varkappa_1, \varkappa_2)\right\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2 \leq \frac{3}{4})} = sup\left(\iint_{\varkappa_1+\varkappa_2 \leq \frac{3}{4}} \left|\varkappa_1 \frac{\partial^2}{\partial^2 \varkappa_2} \mathcal{M}_n f\lambda_n(\varkappa_1, \varkappa_2)\right|^p d\varkappa_2 d\varkappa_1\right)^{\frac{1}{p}}$$

$$= sup\left(\iint_{\varkappa_1+\varkappa_2 \leq \frac{3}{4}} \left|\varkappa_1 \varkappa_1^{-1}(1-\varkappa_1-\varkappa_2)^{-1} \sum_{k+m \leq n} Q_{n,k,m}(\varkappa_1, \varkappa_2) \mathrm{F}_{\mathrm{n,k,m}} \lambda_n\right|^p d\varkappa_2 d\varkappa_1\right)^{\frac{1}{p}}$$

Such that

$$Q_{n,k,m}(\varkappa_1, \varkappa_2) = \varkappa_1^{-1}(1-\varkappa_1-\varkappa_2)^{-1} \wp_{n,k,m}(\varkappa_1, \varkappa_2)$$
$$\left(k(k-1)(1-\varkappa_1-\varkappa_2)^2 - 2k(n-k-m)\varkappa_1(1-\varkappa_1-\varkappa_2) + (n-k-m)\right)(n-k-m-1){\varkappa_1}^2$$

Let $\varkappa_1 + \varkappa_2 \leq \frac{3}{4}$, then by Holder inequality and (2, 1) we have

$$\leq sup\left(\iint_{\varkappa_1+\varkappa_2 \leq \frac{3}{4}} (1-\varkappa_1-\varkappa_2)^{-1} \sum_{k+m \leq n} \left|Q_{n,k,m}(\varkappa_1, \varkappa_2)\right|^{\frac{1}{q}} \sum_{k+m \leq n} \left|Q_{n,k,m}(\varkappa_1, \varkappa_2)\right|^{\frac{1}{p}}\right.$$
$$\left.\left|\mathrm{F}_{\mathrm{n,k,m}} \lambda_n\right|^1 d\varkappa_2 d\varkappa_1\right)^{\frac{1}{p}} \quad (2, 7)$$

For k, m≥ 1 and k+m≤ n, therefore we need to estimate the sum in (2, 1).

For $\varkappa_1 + \varkappa_2 \leq \frac{3}{4}$ we can obtain

$$(1-\varkappa_1-\varkappa_2)^{-1} \sum_{k+m \leq n} Q_{n,k,m}(\varkappa_1, \varkappa_2)$$
$$= \sum_{k+m \leq n} (1-\varkappa_1-\varkappa_2)^{-1} \varkappa_1^{-1} (1-\varkappa_1-\varkappa_2)^{-1} \wp_{n,k,m}(\varkappa_1, \varkappa_2)[k(k-1)(1-\varkappa_1-\varkappa_2)^2$$
$$- 2k(n-k-m)\varkappa_1(1-\varkappa_1-\varkappa_2) + (n-k-m)(n-k-m-1){\varkappa_1}^2]$$
$$= \sum_{k+m \leq n} n^2 (1-\varkappa_1-\varkappa_2)^{-2} \varkappa_1^{-1} \left(\wp_{n,k,m}(\varkappa_1, \varkappa_2)\frac{k^2}{n^2}(1-\varkappa_1-\varkappa_2)^2 - \right.$$
$$\wp_{n,k,m}(\varkappa_1, \varkappa_2)\frac{k}{n^2}(1-\varkappa_1-\varkappa_2)^2 - 2\frac{k}{n^2}(n-k-m)\varkappa_1(1-\varkappa_1-\varkappa_2)$$
$$\wp_{n,k,m}(\varkappa_1, \varkappa_2) + \wp_{n,k,m}(\varkappa_1, \varkappa_2)(n-k-m)^2 {\varkappa_1}^{-2} n^{-2} + (n-k-m)n^{-2}{\varkappa_1}^{-2}$$
$$\wp_{n,k,m}(\varkappa_1, \varkappa_2)$$

$= n^2(1-\varkappa_1-\varkappa_2)^{-2}{\varkappa_1}^{-1}B_n(s^2(1-\varkappa_1-\varkappa_2)^2-2s(1-s-t)\varkappa_1(1-\varkappa_1-\varkappa_2)$

$+(1-s-t)^2{\varkappa_1}^2$

$+(s(1-\varkappa_1-\varkappa_2)^2+(1-s-t){\varkappa_1}^2/n,\varkappa_1,\varkappa_2)$

$B_n(s^2(1-\varkappa_1-\varkappa_2)^2-2s(1-s-t)\varkappa_1(1-\varkappa_1-\varkappa_2)$

$+(1-s-t)^2{\varkappa_1}^2+$

$(s(1-\varkappa_1-\varkappa_2)^2+(1-s-t){\varkappa_1}^2/n,\varkappa_1,\varkappa_2)$

$\leq 4^p n^p {\varkappa_1}^p$

The other two sums in (2,1) can be estimated in the same way thus:

$$\left\|\varkappa_1\frac{\partial^2}{\partial^2\varkappa_2}\mathcal{M}_n f\lambda_n(\varkappa_1,\varkappa_2)\right\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\leq\frac{3}{4})}$$

$$\leq 4^p(5n)^{\frac{1}{q}}\left(\sup\iint\sum_{k+m\leq n}\left|Q_{n,k,m}(\varkappa_1,\varkappa_2)\right|^1 d\varkappa_2 d\varkappa_1(n+1)(n+2)\right.$$

$$\left.\left(\iint\wp_{n,k,m}(s,t)dsdt\right)^{\frac{p}{q}}\iint\wp_{n,k,m}(s,t)|f\lambda_n(s,t)|^p dsdt\right)^{\frac{1}{p}}$$

$$\leq 4^p(5n)^{p-1}\left(\frac{2}{n}\right)(n+1)(n+2)\iint\sum\nolimits_{k+m\leq n}\wp_{n,k,m}(s,t)|f\lambda_n(s,t)|^p\,dsdt$$

$$\leq (80)^p(n)^p\|f\|_{p,\lambda_n}$$

in the same way

$$\left\|(\varkappa_1\varkappa_2)^{\frac{1}{2}}\frac{\partial^2}{\partial^2\varkappa_2}\mathcal{M}_n f\lambda_n(\varkappa_1,\varkappa_2)\right\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\leq\frac{3}{4})}\leq(80)^p(n)^p\|f\|_{p,\lambda_n}\cdots \quad (2, 8)$$

3.Main results:

Theorem (3.1):

If $f\in L_{p,\lambda_n}(\mathfrak{H})$, then there is a positive constant C, such that:

$$\|\mathcal{M}_{n,2}f(x,y)-f(x,y)\|_{L_{p\lambda_n},\varkappa_1+\varkappa_2\leq\frac{2}{3}}\leq C\left\{\frac{\Omega_0 f+\|f\|_{p,\lambda_n}}{n}\right\} \quad (3, 1)$$

Proof:

Assume that $f(\varkappa_1,\varkappa_2)=0$ in $\varkappa_1+\varkappa_2\geq\frac{3}{4}$

as$f_1(\varkappa_1)=f(\varkappa_1)$in $\varkappa_1+\varkappa_2<\frac{3}{4}$ and $f_1(\varkappa_1)=0$in $\varkappa_1+\varkappa_2\geq\frac{3}{4}$

Let $E_1'=\left\{(\varkappa_1,\varkappa_2):\varkappa_1\geq\frac{1}{4}\right\}$, $E_1'=\left\{(\varkappa_1,\varkappa_2):\varkappa_2\geq\frac{1}{4}\right\}$,

letϑ_i be a partition of unity on $\mathfrak{H}$satisfying the following conditions

$\vartheta_i\in L_{p\lambda_n}$, $\vartheta_i\geq 0$, $\sum_{i==1}^{3}\vartheta_i(\varkappa_1,\varkappa_2)=1$ on $\mathfrak{H}$ and sup$\vartheta_i\subset E_i'$

We shall prove

$$\|\mathcal{M}_{n,2}f(\varkappa_1,\varkappa_2)-f(\varkappa_1,\varkappa_2)\|_{L_{p\lambda_n},\varkappa_1+\varkappa_2\leq\frac{2}{3}}$$

$$\leq C\left\{\frac{\sum_{i=1}^{3}\Omega_i f+\|f\|_{p,\lambda_n}}{n}\right\}\ldots(3, 2)$$

$$\mathcal{M}_{n,2} f\lambda_n(\varkappa_1) = \sum_{i=1}^{3} \mathcal{M}_{n,2}(f\lambda_n\vartheta_i)(\varkappa_1), f = \sum_{i=1}^{3} f\vartheta_i \ on\ \mathfrak{H}$$

We first consider the term $f\vartheta_3$ since $\vartheta_3\,(\varkappa_1,\varkappa_2) = 0$

On the complement of $\grave{E}_3$we can assume that $f(\varkappa_1,\varkappa_2) = 0$

$$on\grave{E}_3{}^{*} = \left\{(\varkappa_1,\varkappa_2): x,y \geq 0, \varkappa_1 + \varkappa_2 > \tfrac{4}{5}\right\} \quad (3, 3)$$

Define $f_s(t) = f(s,(1-s)\,t)$ for s, t∈ [0,1] and since $\wp_{n,k,m}(\varkappa_1,\varkappa_2)$=$\wp_{n,k}(\varkappa_1)\wp_{n-k,m}\left(\frac{\varkappa_2}{1-\varkappa_1}\right)$

then from formula (2, 10) we get:

$$\mathcal{M}_{n,2} f\lambda_n\vartheta_3\,(\varkappa_1,\varkappa_2) - f\lambda_n\vartheta_3\,(\varkappa_1,\varkappa_2) = \mathcal{M}_{n,2}(f_1\lambda_n\vartheta_3 - f\lambda_n\vartheta_3 - f\lambda_n\vartheta_3 \left(s,(1-s)\tfrac{\varkappa_2}{1-\varkappa_1}\right) + f\lambda_n\vartheta_3\left(s,(1-s)\tfrac{\varkappa_2}{1-\varkappa_1}\right)(\varkappa_1,\varkappa_2)$$

$$=(n+2)(n+1)\sum_{k=0}^{n}\sum_{m=0}^{n-k}\wp_{n,k,m}(\varkappa_1,\varkappa_2)\iint \wp_{n,k,m}(s,t)\,(f_1\lambda_n\vartheta_3\,(s,t) -f\lambda_n\vartheta_3\left(s,(1-s)\tfrac{\varkappa_2}{1-\varkappa_1}\right) + f\lambda_n\vartheta_3\left(s,(1-s)\tfrac{\varkappa_2}{1-\varkappa_1}\right) - f\lambda_n\vartheta_3\,(\varkappa_1,\varkappa_2))dsdt$$

$$= \sum_{k=0}^{n}\wp_{n,k}(\varkappa_1)(n+2)\sum_{m=0}^{n-k}(n+1)\,\wp_{n-k,m}\left(\tfrac{\varkappa_2}{1-\varkappa_1}\right)\iint \wp_{n,k,m}(s,t)\,f\lambda_n\vartheta_3 (s,t) - f\lambda_n\vartheta_3\left(s,(1-s)\tfrac{\varkappa_2}{1-\varkappa_1}\right) + f\lambda_n\vartheta_3\left(s,(1-s)\tfrac{\varkappa_2}{1-\varkappa_1}\right) - f\lambda_n\vartheta_3\,(\varkappa_1,\varkappa_2))dsdt$$

$$= \sum_{k=0}^{n}\wp_{n,k}(\varkappa_1)(n+2)\sum_{m=0}^{n-k}(n+1)\,\wp_{n-k,m}\left(\tfrac{\varkappa_2}{1-\varkappa_1}\right)\int_0^1\int_0^{1-s}\wp_{n,k}(s)\wp_{n-k,m}\,\tfrac{t}{(1-s)}\left(f\lambda_n\vartheta_3\,(s,t) - f\lambda_n\vartheta_3\left(s,(1-s)\tfrac{\varkappa_2}{1-\varkappa_1}\right) + f\lambda_n\vartheta_3\left(s,(1-s)\tfrac{\varkappa_2}{1-\varkappa_1}\right) - f\lambda_n\vartheta_3\,(\varkappa_1,\varkappa_2)\right)dsdt$$

$$= \sum_{k=0}^{n}\wp_{n,k}(\varkappa_1)(n+2)\int_0^1\wp_{n,k}(s)\sum_{m=0}^{n-k}(n+1)\,\wp_{n-k,m}\left(\tfrac{\varkappa_2}{1-\varkappa_1}\right)\int_0^{1-s}\Big(\wp_{n-k,m}\,\tfrac{t}{(1-s)}\left(f\lambda_n\vartheta_3\,(s,t) - f\lambda_n\vartheta_3\left(s,(1-s)\tfrac{\varkappa_2}{1-\varkappa_1}\right) + f\lambda_n\vartheta_3\left(s,(1-s)\tfrac{\varkappa_2}{1-\varkappa_1}\right)dt\right)ds$$

$$+ \sum_{k=0}^{n}\wp_{n,k}(\varkappa_1)(n+2)\int_0^1\Big(\wp_{n,k}(s)\left(\sum_{m=0}^{n-k}(n+1)\,\wp_{n-k,m}\left(\tfrac{\varkappa_2}{1-\varkappa_1}\right)\right. \int_0^{1-s}\Big(\wp_{n-k,m}\,\tfrac{t}{(1-s)}dt\Big)\left(f\lambda_n\vartheta_3\left(s,(1-s)\tfrac{\varkappa_2}{1-\varkappa_1}\right)f\lambda_n\vartheta_3\,(\varkappa_1,\varkappa_2)\right)ds$$

$$= \sum_{k=0}^{n}\wp_{n,k}(\varkappa_1)(n+2)\int_0^1\left\{\wp_{n+1,k}(s)\left(M_{n-k}\,(f_s\lambda_n\vartheta_3\,,\frac{\varkappa_2}{1-\varkappa_1} - f_s\lambda_n\vartheta_3\,\frac{\varkappa_2}{1-\varkappa_1}\right)\right\}ds + M_{n-k}^{*}\,(f_s\lambda_n\vartheta_3\,(.,(1-.)\,\tfrac{\varkappa_2}{1-\varkappa_1},x) - f_s\lambda_n\vartheta_3\,(x,y)) \quad (3, 4)$$

$$= I + J$$

Let$I = \sum_{k=0}^{n}\wp_{n,k}(\varkappa_1)(n+2)\int_0^1\left\{\wp_{n+1,k}(s)\left(M_{n-k}\,(f_s\lambda_n\vartheta_3\,,\frac{\varkappa_2}{1-\varkappa_1} - f_s\lambda_n\vartheta_3\,\frac{\varkappa_2}{1-\varkappa_1}\right)\right\}ds$

Then by taking the multiplier norm and *By Holder inequality and Fubini`s theorem, we get*

$$\|I\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\leq\frac{2}{3})} \leq sup\int_0^{\frac{2}{3}}\left\{\sum_{k=0}^{n}\wp_{n,k}(\varkappa_1)\,(1-\varkappa_1)(n+2)\int_0^1\wp_{n+1,k}(s)\right. \left(\int_0^1 |M_{n-k}\,(f_s\lambda_n\vartheta_3\,,y/(1-x)f_s\lambda_n\vartheta_3\ y/(1-x))\}|^{p}\right)$$

$$\left(\iint_{\varkappa_1+\varkappa_2\leq\frac{2}{3}}\left(\left|\sum_{k=0}^{n}\wp_{n,k}(\varkappa_1)\int_0^1\{\wp_{n+1,k}(s)(M_{n-k}\,(f_s\lambda_n\vartheta_3\,,z - f_s\lambda_n\vartheta_3\,z)\}\right|^{p}dz\right)dsd\varkappa_1\right)^{1/p}$$

From lemma (2, 1) we get:

$$sup\int_0^1 |M_{n-k}\,(f_s\lambda_n\vartheta_3\,,z - f_s\lambda_n\vartheta_3\,z|^{p}dz$$

$$\leq C_p\left(\|f_s\|_{\mathrm{p},\lambda_\mathrm{n}} + \left\|\mathrm{z}(1-\mathrm{z})f_s{}''\vartheta_3\,z\right\|_{\mathrm{p},\lambda_\mathrm{n}}\right)(n-k+1)^{-p} \quad (3, 5)$$

Where C_p is a constant independent of n, k and$f_s\lambda_n\vartheta_3$

$$\|I\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\le\frac{2}{3})} \le sup\left(\int_0^{\frac{2}{3}}\left|\left\{\sum_{k=0}^{n}\wp_{n,k}(\varkappa_1)\,(n\right.\right.\right.$$

$$\left.+2)\int_0^1 \wp_{n+1,k}(s)\,C_p(n-k+1)^{-p} sup\left(\int_0^1 |f_s\lambda_n\vartheta_3\,(s,(1-s)z)|^p\right)^{\frac{1}{p}} dz$$

$$\left.\left.\left.+\,sup\int_0^1\left(\left|z(1-z)(1-s)^2\left(\frac{\partial^2 f}{\partial y^2}\lambda_n\vartheta_3\right)(s,(1-s)z\right|^p dz\right)^{\frac{1}{p}} ds\right\}dx\right|\right)^{\frac{1}{p}}$$

$$\le C_p sup\sum_{k=0}^{n}\int_0^{\frac{2}{3}}\wp_{n,k}(\varkappa_1)\,(n+2)\int_0^1\wp_{n+1,k}(s)(n-k+1)^{-p}$$

$$sup\int_0^{1-s}\left(\left|f\lambda_n\vartheta_3\,(s,\varkappa_2)\varkappa_2(1-s-\varkappa_2)\left(\frac{\partial^2 f}{\partial y^2}\lambda_n\vartheta_3\right)(s,\varkappa_2)\right|^p d\varkappa_2\,ds\right)^{\frac{1}{p}}$$

$$\le C_p(n+2)sup\int_0^1\left|\sum_{k=0}^{n}\wp_{n,k}(s)\,(n-k+1)^{-p-1} sup\int_0^{1-s}\left(|f\lambda_n\vartheta_3\,(s,\varkappa_2)|^p d\varkappa_2\right)^{\frac{1}{p}}\right.$$

$$\left.+\,sup\int_0^{1-s}\left|\varkappa_2(1-s-\varkappa_2)\left(\frac{\partial^2 f}{\partial y^2}\lambda_n\vartheta_3\right)(s,\varkappa_2)\right|^p d\varkappa_2\,ds\right|^{\frac{1}{p}}$$

By [3] let $m > p + 1$dependent on p then

$$\sum_{k=0}^{n}\wp_{n,k}(s)(n-k+1)^{-m} \le m!(1-s)^{-m} \qquad (3, 6)$$

By assumption $f\lambda_n\vartheta_3\,{}_{s+\varkappa_2\ge\frac{2}{3}} = 0$, we obtain

$$\|I\|_{L_{p\lambda_n}(s+\varkappa_2\le\frac{2}{3})} \le C_p'(n+2)^{\frac{1}{p}}$$

$$sup\int_0^1\left|\left|\frac{\sum_{k=0}^{n}\wp_{n,k}(s)\,n}{(n-k+1)^m}\right|^{\frac{p+1}{m}} n^{-p-1} sup\int_0^{1-s}\left(|f\lambda_n\vartheta_3\,(s,y)|^p dy\right)^{\frac{1}{p}}\right.$$

$$\left.+\,sup\int_0^{1-s}\left|\varkappa_2(1-s-\varkappa_2)\left(\frac{\partial^2 f}{\partial y^2}\lambda_n\vartheta_3\right)(s,\varkappa_2)\right|^p d\varkappa_2\,ds\right|^{\frac{1}{p}}$$

$$\le C_p'(n+2)n^{-p-1}(1-s)^{-p-1}(m!)^{\frac{p+1}{m}}$$

$$\mathrm{sup}\left(\iint_{s+y\le\frac{3}{4}}\left(|f\lambda_n\vartheta_3\,(s,\varkappa_2)|^p d\varkappa_2\right)^{\frac{1}{p}} + \int_0^{1-s}\left|\varkappa_2(1-s-\varkappa_2)\left(\frac{\partial^2 f}{\partial {\varkappa_2}^2}\lambda_n\vartheta_3\right)(s,\varkappa_2)\right|^p ds\,d\varkappa_2\right)^{\frac{1}{p}}$$

$$\le C_p^{`}(n+2)n^{-p}\left(\|f\vartheta_3\|_{\mathrm{p},\lambda_\mathrm{n}} + \Omega_i\|f\vartheta_3\|_{\mathrm{p},\lambda_\mathrm{n}}\right)\ldots(3, 7)$$

Where $C_p^{`}$ is independent of f and n, to get J from lemma (2.1) we get

$$\|J\|_{L_{p\lambda_n}(s+\varkappa_2\le\frac{2}{3})} \le sup\left(\int_0^{\frac{2}{3}} d\varkappa_1\int_0^{\frac{2}{3}}|M_n^*(f\lambda_n\vartheta_3\,(.,(1-.)z,\varkappa_1) - f\lambda_n\vartheta_3\,(\varkappa_1(1-\varkappa_1)z|^p\right)^{1/p}$$

$$\le (2A)\mathrm{sup}\left(\int_0^{\frac{3}{4}} dz\left\{(n+1)^{-1}\left[sup\int_0^1|f\lambda_n\vartheta_3\,(\varkappa_1(1-\varkappa_1)z\,|^p\, d\varkappa_1\right]\right\}\right)^{\frac{1}{p}}$$

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$$\left(+3\sup\left(\int_0^1 \left|\varkappa_1(1-\varkappa_1)\left(\frac{\partial^2 f}{\partial x^2}\lambda_n\vartheta_3\right)(\varkappa_1(1-\varkappa_1)z\right|^p d\varkappa_1\right)^{\frac{1}{p}} + sup\left(\int_0^1 \left|\varkappa_1(1-\varkappa_1)z^2\left(\frac{\partial^2 f}{\partial y^2}\lambda_n\vartheta_3\right)(\varkappa_1(1-\varkappa_1)z)\right|^p d\varkappa_1\right)^{\frac{1}{p}}\right)^{\frac{1}{p}}$$

$$\leq (2A)(n+1)^{-1}$$

$$\sup\left(\int_0^1\int_0^{1-x}|f\lambda_n\vartheta_3(\varkappa_1,\varkappa_2)|^p/(1-\varkappa_1)d\varkappa_2 d\varkappa_1\right)^{\frac{1}{p}} +(6A)(n+1)^{-1}$$

$$sup\left(\int_0^1\int_0^{1-x}\left|\varkappa_1\left(\frac{\partial^2 f}{\partial {\varkappa_1}^2}\lambda_n\vartheta_3\right)(\varkappa_1,\varkappa_2)\right|^p/(1-\varkappa_1)d\varkappa_2 d\varkappa_1\right)^{\frac{1}{p}} + (6A)(n+1)^{-1}sup$$

$$\left(\int_0^1\int_0^{1-x}\frac{\left|(\varkappa_1\varkappa_2)^{\frac{1}{2}}\left(\frac{\partial^2 f}{\partial\varkappa_1\partial y}\lambda_n\vartheta_3\right)(\varkappa_1,\varkappa_2)\right|^p}{(1-\varkappa_1)d\varkappa_2 d\varkappa_1}\right)^{\frac{1}{p}}$$

$$\leq 4(2A)(n+1)^{-1}\|f\vartheta_3\|_{p,\lambda_n} + 12(6A)(n+1)^{-1}\left(\Omega_i(f\vartheta_3)_{p,\lambda_n}\right)^p$$

Then we get for $f \in W_{p,\lambda_n}$ and $f\lambda_n\vartheta_3|_{s+y\geq b} = 0$

$$\|\mathcal{M}_n f\vartheta_3 - f\vartheta_3\|_{L_{p\lambda_n}(\mathfrak{H})} \leq C_p''\left(\|f\vartheta_3\|_{p,\lambda_n} + \Omega_i\|f\vartheta_3\|_{p,\lambda_n}/n\right)$$

Where C_p''depending only on p.

Now let g= $f\vartheta_3$ and $\zeta = f - g$ then

$$\|\mathcal{M}_n f\vartheta_3 - f\vartheta_3\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\leq\frac{2}{3})} \leq \|\mathcal{M}_n g - g\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\leq\frac{2}{3})} + \|\mathcal{M}_n\zeta\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\leq\frac{2}{3})}$$

$$\leq C_p''\left(\|g\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\leq\frac{2}{3})} + \Omega_i(g)_{L_{p\lambda_n}}\right)/n + \left\|\sum_{k+m\leq n}\wp_{n,k,m}(\varkappa_1,\varkappa_2)(n+1)(n+2)\sum_{k+m\leq n}\wp_{n,k,m}(\varkappa_1,\varkappa_2)\iint\wp_{n,k,m}(s,t)|\zeta(s,t)|\,ds\,dt\right\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\leq\frac{2}{3})} \quad (3,8)$$

Then we obtain from the second term and by using lemma (4,2)[5]

$$\left\|\sum_{k+m\leq n}\wp_{n,k,m}(\varkappa_1,\varkappa_2)(n+1)(n+2)\sum_{k+m\leq n}\wp_{n,k,m}(\varkappa_1,\varkappa_2)\iint\wp_{n,k,m}(s,t)|\zeta(s,t)|\,ds\,dt\right\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\leq\frac{2}{3})}$$

$$\leq M_p n^{-p-3}(n+1)(n+2)(1+\|\vartheta_3\|_{L_{p\lambda_n}})sup\left(\iint\wp_{n,k,m}(s,t)|f(s,t)|^p dsdst\right)^{\frac{1}{p}}$$

$$\leq 8M_p(1+\|\vartheta_3\|_{L_{p\lambda_n}})\|f\|_{p,\lambda_n}/n$$

Again by lemma (2.3) and assumption (1-ϑ_3) $= 0$

$$\left\|\sum_{k+m\leq n}\wp_{n,k,m}(\varkappa_1,\varkappa_2)(n+1)(n+2)\sum_{k+m\leq n}\wp_{n,k,m}(\varkappa_1,\varkappa_2)\iint\wp_{n,k,m}(s,t)|(1-\vartheta_3(s,t))f(s,t)|\,ds\,dt\right\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\leq\frac{2}{3})}$$

$$\leq M_p(1+\|\vartheta_3\|_{L_{p\lambda_n}}n^{-p-3}(n+1)(n+2))\|f\|_{p,\lambda_n}/n$$

$$\leq 8M_p(1+\|\vartheta_3\|_{L_{p\lambda_n}})\|f\|_{p,\lambda_n}/n$$

And for the first term we have to estimate $\Omega_i(g)_{p,\lambda_n}$in terms of $\Omega_i(g)_{p,\lambda_n} + \|f\|_{p,\lambda_n}$

We estimate only $\left\|y\left(\frac{\partial^2}{\partial y^2}\right)g(\varkappa_1,\varkappa_2)\right\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\leq\frac{2}{3})}$

Since the other two terms can be estimated in the same way, we know that:

$$(\vartheta_3)_{x+y\geq b} = 0 \; and \; \frac{\partial}{\partial y}\vartheta_3 = 0$$

Then we

get$\left\|y\left(\frac{\partial^2}{\partial y^2}\right)g(\varkappa_1,\varkappa_2)\right\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\le\frac{2}{3})} = \left\|y\left(\frac{\partial^2}{\partial y^2}\vartheta_3\right)(\varkappa_1,\varkappa_2)f(\varkappa_1,\varkappa_2)+\right.$

$\left.2y\left(\frac{\partial}{\partial y}\vartheta_3\right)(\varkappa_1,\varkappa_2)\left(\frac{\partial}{\partial x}f\right)(\varkappa_1,\varkappa_2)+y\vartheta_3(\varkappa_1,\varkappa_2)\left(\frac{\partial^2}{\partial y^2}f\right)(\varkappa_1,\varkappa_2)\right\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\le\frac{2}{3})}$

$$\le \left\|\frac{\partial^2}{\partial{\varkappa_2}^2}\vartheta_3\right\|_{p,\lambda_n}\|f\|_{p,\lambda_n}+2\left\|\frac{\partial}{\partial\varkappa_1}f\right\|_{p,\lambda_n}\left\|\frac{\partial}{\partial\varkappa_2}\vartheta_3\right\|_{L_{p\lambda_n}}+\|\vartheta_3\|_{L_{p\lambda_n}}\Omega_i(f)_{p,\lambda_n} \quad (3, 9)$$

Now for any $\varkappa_1\in[c,b]$ we define a function

$\zeta_{\varkappa_1}(z)=f\left(\left(\frac{2}{3}-\varkappa_1\right)z,\varkappa_1\right)$and by (Ding-Xuan Zhou, 1990)

$$\|\zeta_{\varkappa_1}^{\sim}\|_{p,\lambda_n[0,1]}=sup\left(\int_0^1\left|\left(\frac{2}{3}-\varkappa_1\right)\frac{\partial}{\partial x}f\left(\left(\frac{2}{3}-\varkappa_1\right)z,x\right)\right|^p dz\right)^{\frac{1}{p}}$$

$$=sup\left(\int_0^{\frac{2}{3}-x}\left(\frac{2}{3}-\varkappa_1\right)^{p-1}\left|\frac{\partial}{\partial y}f(\varkappa_1,\varkappa_2)\right|^p d\varkappa_1\right)^{\frac{1}{p}}$$

$$\le M_p^{`}\|\zeta_{\varkappa_1}^{\sim}\|_{p,\lambda_n}+\|z(1-z)\zeta_{\varkappa_1}^{\sim\sim}(z)\|_{L_{p\lambda_n}}$$

$$=M_p^{`}\left(sup\int_0^{\frac{2}{3}-x}|f\lambda_n(\varkappa_1,\varkappa_2)|^p\left(\frac{2}{3}-\varkappa_1\right)^{-1}d\varkappa_2+sup\int_0^{\frac{2}{3}-\varkappa_1}\left|\varkappa_2\left(\frac{2}{3}-\varkappa_1-\varkappa_2\right)\left(\frac{\partial^2}{\partial y^2}f\lambda_n\right)(\varkappa_1,\varkappa_2)\right|^p\left(\frac{2}{3}-\varkappa_1\right)^{-1}d\varkappa_1\right)^{\frac{1}{p}}$$

Then:

$$\left\|\frac{\partial}{\partial x}f\right\|_{p,\lambda_n(c\le\varkappa_1+\varkappa_2\le b)}$$

$$\le\left(\frac{2}{3}-b\right)^{-1}\left\|\left(\frac{2}{3}-\varkappa_1\right)\left(\frac{\partial}{\partial x}f\right)(\varkappa_1,\varkappa_2)\right\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\le\frac{2}{3})}$$

$$\le\left(\frac{2}{3}-b\right)^{-1}$$

$$sup\left(\int_0^{\frac{2}{3}}\left\{\int_0^{\frac{2}{3}-\varkappa_1}\left|\frac{\partial}{\partial\varkappa_2}f\lambda_n(\varkappa_1,\varkappa_2)\right|^p dy\left(\frac{2}{3}-\varkappa_1\right)^p\right\}d\varkappa_1\right)^{\frac{1}{p}}$$

$$\le\left(\frac{2}{3}-b\right)^{-1}$$

$$sup\left(\int_0^{\frac{2}{3}}M_p^{`}\left(\frac{2}{3}-\varkappa_1\right)\int_0^{\frac{2}{3}-\varkappa_1}|f\lambda_n(\varkappa_1,\varkappa_2)|^p\left(\frac{2}{3}-\varkappa_1\right)^{-p}d\varkappa_2\right.$$

$$\left.+\int_0^{\frac{2}{3}-\varkappa_2}\left|\varkappa_2\left(\frac{\partial^2}{\partial{\varkappa_2}^2}f\lambda_n\right)(\varkappa_1,\varkappa_2)\right|^p\left(\frac{2}{3}-\varkappa_1\right)^{-p}d\varkappa_2 d\varkappa_1\right)^{\frac{1}{p}}$$

$$\le M_p^{`}\left(\frac{2}{3}-b\right)^{-1}\left(\|f\|_{p,\lambda_n}+\Omega_i(f)_{L_{p\lambda_n}}\right)$$

We get from lemma(2.1)

$$\left\|\frac{\partial^2}{\partial y^2}g(\varkappa_1,\varkappa_2)\right\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\le\frac{2}{3})}\le M_p^*\left(\|f\|_{p,\lambda_n}+\Omega_i(f)_{L_{p\lambda_n}}\right)$$

Where $M_p^*$is a constant depending only on p

We have also

$$\Omega_i(f\vartheta_3)_{L_{p\lambda_n}}\le M_p^*\left(\|f\|_{p,\lambda_n}+\Omega_i(f)_{L_{p\lambda_n}}\right) \quad (3, 10)$$

By combining all estimates above with (2.11), we obtain:

$$\|\mathcal{M}_n f-f\|_{L_{p\lambda_n}(\varkappa_1+\varkappa_2\le\frac{2}{3})}\le C_p{}''\left(\frac{\|\vartheta_3\|_{L_{p\lambda_n}}\|f\|_{p,\lambda_n}+M_p^*\left(\|f\|_{p,\lambda_n}+\Omega_i(f)_{L_{p\lambda_n}}\right)}{n}\right.$$

$$\left.+\frac{16M_p\left(1+\|\vartheta_3\|_{L_{p\lambda_n}}\right)\|f\|_{p,\lambda_n}}{n}\right)$$

$$\le C\,\Omega_i(f)_{L_{p\lambda_n}}+\|f\|_{p,\lambda_n}/n$$

Theorem (3.2):

For any p≥ 1 and for any $f \in L_{p\lambda_n}(\mathfrak{H})$ we have

$$\|\mathcal{M}_n f(\varkappa_1,\varkappa_2) - f(\varkappa_1,\varkappa_2)\|_{L_{p\lambda_n}(\mathfrak{H})} \leq C_p K_2\left(f, n^{\frac{-1}{2}}\right)_{p,\lambda_n} + \frac{\|f\|_{p,\lambda_n}}{n} \quad (3, 11)$$

Proof:

let f be any function belonging to $L_{p\lambda_n}(\mathfrak{H})$, p≥ 1and for any g∈ $C^2(\mathfrak{H})$we get

$$\|\mathcal{M}_n f(\varkappa_1,\varkappa_2) - f(\varkappa_1,\varkappa_2)\|_{L_{p\lambda_n}(\mathfrak{H})} = sup\left(\int_{\mathfrak{H}} |\mathcal{M}_n f\lambda_n(\varkappa_1,\varkappa_2) - f\lambda_n(\varkappa_1,\varkappa_2)|^p d\varkappa_1\right)^{1/p}$$

$$= sup\left(\int_{\mathfrak{H}} |\mathcal{M}_n f\lambda_n(\varkappa_1,\varkappa_2) - \mathcal{M}_n g\lambda_n(\varkappa_1,\varkappa_2) + \mathcal{M}_n g\lambda_n(\varkappa_1,\varkappa_2) - g\lambda_n(\varkappa_1,\varkappa_2) + g\lambda_n(\varkappa_1,\varkappa_2) - f\lambda_n(\varkappa_1,\varkappa_2)|^p d\varkappa_1\right)^{1/p}$$

$$\leq sup\left(\int_{\mathfrak{H}} |\mathcal{M}_n f\lambda_n(\varkappa_1,\varkappa_2) - \mathcal{M}_n g\lambda_n(\varkappa_1,\varkappa_2)|^p dx\right)^{1/p} + \sup\left(\int_{\mathfrak{H}} |\mathcal{M}_n g\lambda_n(\varkappa_1,\varkappa_2) - g\lambda_n(\varkappa_1,\varkappa_2)|^p dx\right)^{1/p} + sup\left(\int_{\mathfrak{H}} |f\lambda_n(\varkappa_1,\varkappa_2) - g\lambda_n(\varkappa_1,\varkappa_2)|^p dx\right)^{1/p}$$

$$\leq sup\left(\int_{\mathfrak{H}} \left|\mathcal{M}_n\left(f\lambda_n(\varkappa_1,\varkappa_2) - g\lambda_n(\varkappa_1,\varkappa_2)\right)\right|^p dx\right)^{1/p} + \sup\left(\int_{\mathfrak{H}} |\mathcal{M}_n g\lambda_n(\varkappa_1,\varkappa_2) - g\lambda_n(\varkappa_1,\varkappa_2)|^p dx\right)^{1/p} + sup\left(\int_{\mathfrak{H}} |f\lambda_n(\varkappa_1,\varkappa_2) - g\lambda_n(\varkappa_1,\varkappa_2)|^p dx\right)^{1/p}$$

$$\leq \|\mathcal{M}_n(f-g)(\varkappa_1,\varkappa_2)\|_{L_{p\lambda_n}(\mathfrak{H})} + \|\mathcal{M}_n g(\varkappa_1,\varkappa_2) - g(\varkappa_1,\varkappa_2)\|_{L_{p\lambda_n}(\mathfrak{H})} + \|f(\varkappa_1,\varkappa_2) - g(\varkappa_1,\varkappa_2)\|_{L_{p\lambda_n}(\mathfrak{H})}$$

$$\leq \|f(\varkappa_1,\varkappa_2) - g(\varkappa_1,\varkappa_2)\|_{L_{p\lambda_n}(\mathfrak{H})} + \|\mathcal{M}_n g(\varkappa_1,\varkappa_2) - g(\varkappa_1,\varkappa_2)\|_{L_{p\lambda_n}(\mathfrak{H})} + \|f(\varkappa_1,\varkappa_2) - g(\varkappa_1,\varkappa_2)\|_{L_{p\lambda_n}(\mathfrak{H})}$$

$$\leq 2\|f(\varkappa_1,\varkappa_2) - g(\varkappa_1,\varkappa_2)\|_{L_{p\lambda_n}(\mathfrak{H})} + \|\mathcal{M}_n g(\varkappa_1,\varkappa_2) - g(\varkappa_1,\varkappa_2)\|_{L_{p\lambda_n}(\mathfrak{H})}$$

By using theorem (3.1) we get the quantity is bounded by:

$$\frac{\sum_{i=1}^{3}\|\Omega_i^2 g\|_{p,\lambda_n} + \|f\|_{p,\lambda_n}}{n} + 2\|\mathcal{M}_n g(\varkappa_1,\varkappa_2) - g(\varkappa_1,\varkappa_2)\|_{L_{p\lambda_n}(\mathfrak{H})} + \|f(\varkappa_1,\varkappa_2) - g(\varkappa_1,\varkappa_2)\|_{L_{p\lambda_n}(\mathfrak{H})}$$

then

$$\|\mathcal{M}_n f(\varkappa_1,\varkappa_2) - f(\varkappa_1,\varkappa_2)\|_{L_{p\lambda_n}(\mathfrak{H})} \leq K_2\left(f, n^{\frac{-1}{2}}\right)_{p,\lambda_n} + \frac{\|f\|_{p,\lambda_n}}{n}.$$

Conclusion: the degree of best multiplier approximation of 2-dim. Multiplier Bernstein- Durrmeyer operators in terms of multiplier K-functional.

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