Australian Journal of Basic and Applied Sciences, 11(16) December 2017, Pages: 17-24

AUSTRALIAN JOURNAL OF BASIC AND APPLIED SCIENCES

ISSN:1991-8178
EISSN: 2309-8414

DOI: 10.22587/ajbas.2017.11.16.3
Journal home page: www.ajbasweb.com

Proposed Algorithm for Gumbel Distribution Estimation

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ARTICLE INFO

Article history:
Received 12 October 2017
Accepted 22 December 2017
Available online 15 January 2018

Keywords:
Gumbel Distribution, Maximum Likelihood, Jackknife, Artificial intelligence, Hybrid methods, Simulation.

ABSTRACT

Gumbel distribution was dealt with great care by researchers and statisticians. There are traditional methods to estimate two parameters of Gumbel distribution known as Maximum Likelihood, the Method of Moments and recently the method of re-sampling called (Jackknife). However, these methods suffer from some mathematical difficulties in solving them analytically. Accordingly, there are other non-traditional methods, like the principle of the nearest neighbors, used in computer science especially, artificial intelligence algorithms, including the genetic algorithm, the artificial neural network algorithm, and others that may to be classified as meta-heuristic methods. Moreover, this principle of nearest neighbors has useful statistical features. The objective of this paper is thus to propose a new algorithm where it allows getting the estimation of the parameters of Gumbel probability distribution directly. Furthermore, it overcomes the mathematical difficulties in this matter without need to the derivative of the likelihood function. Taking simulation approach under consideration as empirical experiments where a hybrid method performs optimization of these three traditional methods. In this regard, comparisons have been done between the new proposed method and each pair of the traditional methods mentioned above by efficiency criterion Root of Mean Squared Error (RMSE). As a result, (36) experiments of different combinations of initial values of two parameters (λ: shift parameter and θ: scale parameter) in three values that take four different sample sizes for each experiment. To conclude, the proposed algorithm showed its superiority in all simulation combinations associated with all sample sizes for the two parameters (λ and θ). In addition, the method of Moments was the best in estimating the shift parameter (λ) and the method of Maximum Likelihood was in estimating the scale parameter (θ).

INTRODUCTION

Much attention has been given to the application of Gumbel distribution because of its importance as known "the extreme value threshold" which utilized in global warming, rain falls and dust storms. Gumbel, E.J., (1954) was the first who dealt with this distribution behavior. The class of continuous distributions Extreme value has two types of Extreme value distributions according to (maxima or minima) that dealt with these situations. Gumbel (Type I) distribution is the maximum value that referred to in this paper. The motivation of this paper is that limited researches that dealt with the derivative difficulties associated with this type of distribution in using Maximum Likelihood. Moreover, the slowly resampling procedure in Jackknife Method indicated as disadvantage. The need to estimate both parameters of Gumbel distribution in a modern fast method based on seeking an optimal estimation as a target point. This approach done by reduction the distance between the traditional estimates from the target value which make the achievement to the required estimate faster and slightly.

To Cite This Article: Omar Abdulmohsin Ali., Proposed Algorithm for Gumbel Distribution Estimation. ***Aust. J. Basic & Appl. Sci.,*** *11(16): 17-24, 2017*

MATERIALS AND METHODS

2.1 Gumbel Distribution:

Al-Adilee and Mohamed (2014) argued Gumbel distribution as one of the Extreme Value distributions such as type II - Frechet distribution and type III – Weibull distribution.

The random variable y said to have Gumbel distribution with two parameters λ which represents the shift parameter and θ that represents the scale parameter. If it has the following right skewed probability density function as presented by Aydin and Şenoğlu (2015):

$$f(y;\lambda,\theta)=\frac{e^{-\left(\frac{y-\lambda}{\theta}\right)}}{\theta}\exp\left[-e^{-\left(\frac{y-\lambda}{\theta}\right)}\right] \quad \text{; where is:} \quad y,\lambda\in R \quad , \quad \theta>0 \tag{1}$$

Aydinand Şenoğlu (2015) obtained the Cumulative Probability Function as:

$$F(y)=\Pr(Y\le y)=\exp\left[-e^{-\left(\frac{y-\lambda}{\theta}\right)}\right] \tag{2}$$

2.2 Maximum Likelihood Method (ML):

The method of maximum likelihood has been acknowledged by Raynal-Villasenor (2012) as one of the best methods for parameter estimation of probability distribution functions. The properties of its estimators like the invariance property. The maximum likelihood function can be given as described by Mahdi *et al* (2005).

$$L(y;\lambda,\theta)=\left(\frac{1}{\theta}\right)^n e^{-\sum_{i=1}^{n}\left(\frac{yi-\lambda}{\theta}\right)}\exp\left[-\sum_{i=1}^{n}e^{-\left(\frac{yi-\lambda}{\theta}\right)}\right] \tag{3}$$

But, after taking logarithm:

$$\ln L(y;\lambda,\theta)=-n\ln(\theta)-\sum_{i=1}^{n}\left(\frac{y_i-\lambda}{\theta}\right)-\sum_{i=1}^{n}\exp\left(\frac{y_i-\lambda}{\theta}\right)$$

with partial derivatives below.

$$\frac{\partial\ln L}{\partial\lambda}=\lambda+\theta\ln\left[\frac{\sum_{i=1}^{n}\exp\left(-\frac{y_i}{\theta}\right)}{n}\right]$$

$$\frac{\partial\ln L}{\partial\lambda}=0 \tag{4}$$

$$\rightarrow\hat{\lambda}_{ML}=-\theta\ln\left[\frac{\sum_{i=1}^{n}\exp\left(-\frac{y_i}{\theta}\right)}{n}\right]$$

Or: $$\hat{\lambda}_{ML}=-\frac{\theta}{n}\ln\left[\sum_{i=1}^{n}\exp\left(-\frac{y_i}{\theta}\right)\right] \tag{5}$$

$$\frac{\partial\ln L}{\partial\theta}=\sum_{i=1}^{n}y_i\exp\left(\frac{y_i}{\theta}\right)-(\bar{y}+\lambda)\exp\left(\frac{\lambda}{\theta}\right)$$

$$\frac{\partial\ln L}{\partial\theta}=0 \tag{6}$$

$$\rightarrow \hat{\theta}_{ML} = \bar{y} - \frac{\sum_{i=1}^{n} y_i \exp\left(\frac{y_i}{\hat{\theta}}\right)}{\sum_{i=1}^{n} \exp\left(\frac{y_i}{\hat{\theta}}\right)} \tag{7}$$

Since equations (5) & (7) are very difficult to be solved, hence, iterative numerical Newton-Raphson technique is required in this case to get $\left(\hat{\lambda}_{ML} \quad , \quad \hat{\theta}_{ML}\right)$ respectively.

2.3 Method of Moments (MOM):

The population moment of order **r** can be obtained by Mahdi and Cenac (2005):

$$\mu_r = E\left(y^r\right) \tag{8}$$

While the sample moment of order **r** is:

$$m_r = \frac{1}{n}\sum_{i=1}^{n}\left(y_i^r\right) \tag{9}$$

Therefore, the first moment will be:

$$\mu_1 = E(y) = \lambda + \gamma\theta$$

Where is : $\gamma = 0.5772$ Euler-Mascheroni constant as referred to by Clarke (2002) &Yousef and Sameer (2014).

$$\frac{\sum_{i=1}^{n} y_i}{n} = E\left(y\right) \quad \rightarrow \quad \bar{y} = \lambda + \gamma\theta$$

$$\therefore \lambda = \bar{y} - \gamma\theta$$

Moreover, the second moment will be:

$$Var(y) = E\left(y^2\right) - \left[E(y)\right]^2 = \frac{\pi^2\theta^2}{6}$$

Where is : $\gamma = 0.5772$ Euler's constant.

$$\frac{\sum_{i=1}^{n} y_i^2}{n} = E\left(y^2\right) \quad \rightarrow \quad \frac{\sum_{i=1}^{n} y_i^2}{n} = Var(y) + \left[E(y)\right]^2$$

$$\therefore \frac{\sum_{i=1}^{n} y_i^2}{n} = \frac{\pi^2\theta^2}{6} + \left[\lambda + \gamma\theta\right]^2$$

Where is : $\pi = \frac{22}{7}$

By solving the last equation above we get the moment estimation of (θ) is:

$$\hat{\theta}_{MOM} = \frac{\sqrt{6}}{\pi} S \tag{10}$$

S: the standard deviation of the data. The moment estimation of (λ) is:

$$\hat{\lambda} = \bar{y} - \gamma \frac{\sqrt{6}}{\pi} S$$
$$\rightarrow \hat{\lambda}_{MOM} = \bar{y} - 0.449S \qquad (11)$$

2.4 Jackknife Method (Jack.):

This method is based on successive elimination and repetitions of the values of the random variable in order to get the partial samples that obtained from that elimination. This method is described by Eforn (1979) as "bootstrap method". Hall *et al* (2004) assumed that there is random sample of size (n) of the observations follows random variable which distributed as Gumbel distribution, then many partial samples can be formed, each of them consists of a number of observations so that:

y_{ij}; i = 1, 2, 3, …, n – k ; j = 1, 2, 3, …, m

Where is (k): represents the number of the eliminating observations at each time.

(n-k): the partial sample size resulting after elimination.

(m): the number of partial samples occurred.

$$\hat{\varphi}_{Jack} = n\hat{\varphi}_0 - (n-1)\sum_{i=1}^{m} \frac{\hat{\varphi}_{(i)0}}{m} \qquad (12)$$

Where is: $\hat{\varphi}_0$ is the initial value got from some estimation method.

$\hat{\varphi}_{(i)0}$: the initial value got from some estimation method at (i^{th}) iteration.

2.5 The Proposed Method (Prop.):

Iterative meta-heuristic algorithms are the most rapid widespread modern approaches with computer aided in the world.

The core idea is how to find wide verities of point estimations at once which need only to minimize (or maximize) one objective function (say minimize RMSE) rather than derivatives. The second important thing is how to find a sub-set of the possible estimates for the parameters applied to all possible combinations resulting from the initial estimations. After that all the estimates are compared on the basis of the comparison criterion to determine the optimal combination of estimates and then re-combination approach repeated for a number of times until getting the suitable alternatives which determine the required estimate.

Step 1: Preparing initial values for the both pair parameters (λ , θ) form each of the traditional methods previously obtained , i.e., ML, MOM and Jackknife as "inputs".

Step 2: Distinguish the best estimation method among those traditional estimates (say; λ_0 , θ_o) based on (RMSE) criterion.

Step 3: Conduct a region that formed by huge number of points (say; λ_{0i} , θ_{0i}) ; i=1,2,3, …, 100 that falls into the intervals:

$$\{\lambda_{0i} \in (\lambda_{0i} \pm 0.5) , \theta_{0j} \in (\theta_{0j} \pm 0.5)\} \qquad (13)$$

with all their possible combinations that could be formed.

Step 4: Compare the selected best method (λ_0 ,θ_o) got in step(2) against the (λ_{0i} , θ_{0j}) got in step(3) to obtain the optimal combination among them based on (RMSE) criterion.

Step 5:Continuing subsequent of candidate points iteratively until getting the final estimations $\left(\hat{\lambda}_{prop.}, \hat{\theta}_{prop.}\right)$ by using the stopping rule in terms of Total Squared Error (TSE) as below.

$$TSE = \frac{\left(\hat{\lambda}_i - \hat{\lambda}_{i-1}\right)^2 + \left(\hat{\theta}_i - \hat{\theta}_{i-1}\right)^2}{2} \qquad (14)$$

The optimality will be got at convergence when stopping iff: TSE ≤ 0.0001

Comparison Criterion:

Suppose (φ) be one of the two parameters of Gumbel distribution (λ) or (θ) . Let (rep) be the replications.

Root of Mean Squared Error (RMSE):

$$RMSE = \sqrt{\frac{\sum_{i=1}^{rep} \{\varphi_i - \hat{\varphi}_i\}^2}{rep}} \tag{15}$$

Simulation:

Empirical experiments performed by (**36**) simulations assembled by conducting the combination of parameters as follows:

λ: 2 3 5
θ: 0.5 1.0 2.0
n: 15 30 60 150

Consequently, (y) can be generated as a random variable followed Gumbel distribution with "Inverse Cumulative Distribution Function" by the following steps described by Mohammed (2011):

$$F(y) = \exp\left[-e^{-\left(\frac{y-\lambda}{\theta}\right)}\right]$$

$$\ln[F(y)] = -\exp\left[-\left(\frac{y-\lambda}{\theta}\right)\right]$$

$$\ln\{-\ln[F(y)]\} = -\left(\frac{y-\lambda}{\theta}\right)$$

$$-\ln\{-\ln[F(y)]\} = \frac{y-\lambda}{\theta}$$

Let: U = F(t)
where is U: random variable belongs to [0, 1].
Hence,

$$y = \lambda + \theta\{-\ln[-\ln(U)]\} \tag{16}$$

The numerical results in the following tables: (1, 2, 3 and 4) as below.

Conclusions:

Four estimation methods were compared. The comparison was performed with respect to their (RMSE) criterion. From the previous numerical results obtained from simulation experiments, the following conclusions achieved:

i.) New areas were explored in the estimation space and the proposed algorithm showed its superiority. It was obvious that proposed method is the best method for both parameters based on (RMSE) criterion and all sample sizes.

ii.) (MOM) method got better results gradually as sample size grew for RMSE(λ) from Table (3) compared with (ML) & (Jack). However, it got worse results gradually as sample size grew for RMSE(θ) from Table (4) compared with (ML) &(Jack).

iii.) It can also be noticed from the experimentation results of RMSE for shift parameter (λ) that (MOM) is better than its traditional competitor (ML) & (Jack.) for all sample sizes. Alternatively, the RMSE of scale parameter (θ) results indicated that (ML) is better than (MOM) & (Jack.) for all sample sizes.

iv.) The worst combination of the two estimated parameters was (λ=3 , θ=2) for all sample sizes according to Table (3) and Table (4) regardless to the estimation method.

v.) The best results according to Table (4) for all sample sizes was (θ=0.5) regardless to (λ) value and estimation method.

Table 1: Estimated (λ)

n	λ	θ	Estimation Method			
			ML	MOM	Jack	Prop.
15	2	0.5	2.01687295	2.00013710	1.98485259	2.00000594
15		1.0	2.03526052	2.00795115	1.94935496	2.00381237
15		2.0	2.12073258	2.06315467	1.94771107	2.01882522
15	3	0.5	3.02418095	3.00995118	2.99213706	3.00008071
15		1.0	3.05102813	3.03381198	2.99507406	3.00457684
15		2.0	3.16110025	3.08266404	2.99374980	3.03444466
15	5	0.5	5.02386600	5.00841508	5.00158420	5.00009944
15		1.0	5.03931304	5.00635059	4.96573863	5.00383792
15		2.0	5.12493827	5.05021937	5.03710306	5.01925103
30	2	0.5	2.01163546	2.00631791	1.99867497	1.99998371
30		1.0	2.02226752	2.00755722	1.99785651	2.00044221
30		2.0	2.03146848	2.00917962	1.98271196	2.00394460
30	3	0.5	3.01278058	3.00735583	3.00124696	3.00004512
30		1.0	3.00747531	2.99996731	2.98399509	3.00041554
30		2.0	3.05300313	3.03349474	3.00033677	3.00864270
30	5	0.5	5.00952609	5.00447302	4.99726854	4.99993293
30		1.0	5.03281973	5.01933509	5.00792990	5.00079095
30		2.0	5.03361676	5.01845498	4.98804012	5.00087041
60	2	0.5	2.00504952	2.00202366	1.99961839	1.99997438
60		1.0	2.02249332	2.01733530	2.01132036	2.00016600
60		2.0	2.04498668	2.03467061	2.02264168	2.00252599
60	3	0.5	3.01124667	3.00866765	3.00566064	3.00007537
60		1.0	3.02249333	3.01733530	3.01132108	3.00016662
60		2.0	3.04498666	3.03467061	3.02264021	3.00252503
60	5	0.5	5.01124668	5.00866765	5.00566125	5.00007737
60		1.0	5.02249333	5.01733530	5.01132064	5.00016694
60		2.0	5.04498671	5.03467061	5.02264388	5.00252673
150	2	0.5	2.00205414	2.00224838	1.99982111	1.99995004
150		1.0	2.00793936	2.00560769	2.00338195	1.99994432
150		2.0	2.00706312	2.00341928	1.99838890	2.00024806
150	3	0.5	3.00176578	3.00085482	2.99959749	3.00006455
150		1.0	3.00178400	3.00193642	2.99709763	2.99986241
150		2.0	3.01024694	3.01165291	3.00113711	3.00013413
150	5	0.5	5.00277807	5.00231985	5.00061546	4.99989187
150		1.0	5.00619542	5.00559623	5.00161851	4.99980984
150		2.0	5.01528287	5.01375377	5.00694093	4.99997528

Table 2: Estimated (θ)

n	λ	θ	Estimation Method			
			ML	MOM	Jack	Prop.
15	2	0.5	0.47340057	0.48716625	0.50070369	0.49985106
15		1.0	0.94519179	0.98825839	0.98193606	1.00017895
15		2.0	1.84999516	1.90876550	1.96594206	1.99643857
15	3	0.5	0.46912211	0.48861856	0.49072002	0.49990848
15		1.0	0.92152371	0.98803207	0.99653682	1.00002829
15		2.0	1.91431798	1.92881302	2.01233153	1.99039917
15	5	0.5	0.46899180	0.48389307	0.49623979	0.49992426
15		1.0	0.93386602	0.97064593	0.98903082	1.00022880
15		2.0	1.89079434	1.94936096	2.00435544	1.99033960
30	2	0.5	0.48035508	0.48764823	0.49583896	0.50008038
30		1.0	0.96627305	0.96912808	0.99741166	1.00030171
30		2.0	1.94983898	1.98292174	2.01733912	2.00144285
30	3	0.5	0.47903445	0.48885121	0.49501202	0.50001098
30		1.0	0.96596632	0.99492486	0.99893773	0.99984379
30		2.0	1.91652650	1.96886023	1.97762203	1.99896717
30	5	0.5	0.48757920	0.50025411	0.50385717	0.50009175
30		1.0	0.97170574	0.97919716	1.00504403	1.00003086
30		2.0	1.90885744	1.95988280	1.96999883	1.99997218
60	2	0.5	0.49248828	0.49609373	0.50146617	0.49995777
60		1.0	0.99842427	1.00069664	1.01524060	0.99999819
60		2.0	1.99684850	2.00139328	2.03047637	2.00013115
60	3	0.5	0.49921213	0.50034832	0.50761971	0.49979508
60		1.0	0.99842425	1.00069664	1.01524234	1.00002281
60		2.0	1.99684849	2.00139328	2.03047585	2.00013054
60	5	0.5	0.49921213	0.50034832	0.50761975	0.49979512
60		1.0	0.99842425	1.00069664	1.01523921	1.00002137

60		2.0	1.99684853	2.00139328	2.03047793	2.00013090
150	2	0.5	0.49752254	0.49835525	0.50096611	0.49996120
150		1.0	0.99488753	0.99351863	1.00167836	1.00010323
150		2.0	1.98449844	1.98401702	1.99830001	1.99986705
150	3	0.5	0.49612463	0.49600425	0.49957927	0.49999594
150		1.0	0.99265964	0.99957968	0.99938075	1.00022550
150		2.0	1.98880900	2.00433168	2.00225519	1.99998444
150	5	0.5	0.49574681	0.49821113	0.49933749	0.50008122
150		1.0	0.99518262	1.00318525	1.00212865	0.99994279
150		2.0	1.99018287	1.99748582	2.00389676	2.00012282

Table 3: RMSE(λ)

n	λ	θ	Estimation Method			
			ML	MOM	Jack	Prop.
15	2	0.5	0.12255397	0.10585322	0.14457355	0.00125518
15		1.0	0.24795257	0.22094560	0.27513064	0.00889251
15		2.0	0.51889817	0.46036395	0.57418386	0.09268990
15	3	0.5	0.11831718	0.10386042	0.13873913	0.00120366
15		1.0	0.25002239	0.22397198	0.26923814	0.00861155
15		2.0	0.53366501	0.47365601	0.57070566	0.09854717
15	5	0.5	0.13440976	0.11656101	0.14389626	0.00138880
15		1.0	0.25414928	0.22284453	0.27887802	0.00966722
15		2.0	0.50376716	0.43452776	0.57415456	0.07982590
30	2	0.5	0.07945469	0.07070697	0.08017592	0.00126835
30		1.0	0.16580789	0.15441065	0.16548858	0.00203270
30		2.0	0.32379334	0.29653698	0.33281060	0.02256790
30	3	0.5	0.08463460	0.07882035	0.08423825	0.00128835
30		1.0	0.16925912	0.16004951	0.16921010	0.00160620
30		2.0	0.31546496	0.29472594	0.31499696	0.02736278
30	5	0.5	0.08600126	0.07994527	0.08507069	0.00124837
30		1.0	0.17399038	0.16013726	0.17130431	0.00227011
30		2.0	0.33111669	0.30875702	0.32607332	0.03169727
60	2	0.5	0.05968998	0.05627056	0.05939350	0.00125122
60		1.0	0.11492629	0.10795750	0.11408868	0.00128772
60		2.0	0.22985251	0.21591501	0.22817479	0.00702708
60	3	0.5	0.05746313	0.05397875	0.05704409	0.00112729
60		1.0	0.11492626	0.10795750	0.11408708	0.00128733
60		2.0	0.22985254	0.21591501	0.22817593	0.00702771
60	5	0.5	0.05746314	0.05397875	0.05704514	0.00112762
60		1.0	0.11492628	0.10795750	0.11408791	0.00128801
60		2.0	0.22985259	0.21591501	0.22817880	0.00702704
150	2	0.5	0.03815429	0.03506439	0.03793725	0.00136809
150		1.0	0.07296097	0.06973584	0.07283006	0.00119310
150		2.0	0.14338600	0.13869797	0.14408988	0.00141310
150	3	0.5	0.03584650	0.03467449	0.03602393	0.00125502
150		1.0	0.07130207	0.07001158	0.07194569	0.00123789
150		2.0	0.15808690	0.15117713	0.15705439	0.00142015
150	5	0.5	0.03692508	0.03465282	0.03669940	0.00128951
150		1.0	0.07599905	0.07343580	0.07592677	0.00125237
150		2.0	0.15142838	0.14356337	0.15133244	0.00134335

Table 4: RMSE(θ)

n	λ	θ	Estimation Method			
			ML	MOM	Jack	Prop.
15	2	0.5	0.09280074	0.09991132	0.12707651	0.00120577
15		1.0	0.19909425	0.21130827	0.25449135	0.00162497
15		2.0	0.37783125	0.40150825	0.51122547	0.01736784
15	3	0.5	0.09895411	0.09851459	0.12521339	0.00131140
15		1.0	0.19269626	0.20216291	0.23466462	0.00156591
15		2.0	0.40091586	0.39617763	0.52286084	0.02595584
15	5	0.5	0.10204577	0.10232574	0.12227855	0.00136207
15		1.0	0.18746573	0.20341067	0.26705682	0.00165824
15		2.0	0.39231337	0.41572816	0.51116656	0.02139605
30	2	0.5	0.06192158	0.06784869	0.06183979	0.00122210
30		1.0	0.12624426	0.13825712	0.12612744	0.00157423
30		2.0	0.25996604	0.29525995	0.26735242	0.00586760
30	3	0.5	0.06164153	0.07484130	0.06088505	0.00128101
30		1.0	0.12688509	0.14949160	0.13175902	0.00126936
30		2.0	0.24227728	0.28855446	0.24867299	0.00493893
30	5	0.5	0.06176586	0.07155359	0.06223975	0.00128989

30		1.0	0.12451775	0.14306307	0.12724643	0.00132364
30		2.0	0.24368197	0.28847773	0.24426256	0.00377192
60	2	0.5	0.04344242	0.05237296	0.04416112	0.00126538
60		1.0	0.08298391	0.09552740	0.08548191	0.00131128
60		2.0	0.16596784	0.19105480	0.17096348	0.00146489
60	3	0.5	0.04149197	0.04776370	0.04274120	0.00126634
60		1.0	0.08298395	0.09552740	0.08548643	0.00130859
60		2.0	0.16596782	0.19105480	0.17096200	0.00146630
60	5	0.5	0.04149197	0.04776370	0.04274109	0.00126640
60		1.0	0.08298393	0.09552740	0.08548315	0.00131042
60		2.0	0.16596785	0.19105480	0.17096194	0.00146623
150	2	0.5	0.02722455	0.03436133	0.02730857	0.00133686
150		1.0	0.05330601	0.06470859	0.05259551	0.00123206
150		2.0	0.10922870	0.14078246	0.10918481	0.00126338
150	3	0.5	0.02730716	0.03519561	0.02729417	0.00115416
150		1.0	0.04993126	0.06380849	0.05019086	0.00126576
150		2.0	0.10632348	0.12906069	0.10689809	0.00129757
150	5	0.5	0.02647323	0.03440536	0.02653836	0.00118965
150		1.0	0.05162010	0.06520734	0.05194468	0.00119065
150		2.0	0.10453310	0.12786372	0.10526452	0.00129332

Future Works:

i.) The framework of this paper can also be carried out in the context of other probability distributions for the maxima distribution (e.g. lognormal). Another approach can be updated as a recommendation, by more sophisticated proposition in terms of confidence interval {φ_{0i}±uncertainity points)} instead of the interval (φ_{0i}± fixed points) to emphasize the scope of the research.

ii.) Nonparametric approaches could be used such as smoothing methods: kernel, splines or wavelet methods to make advanced comparison.

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