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Solving the Multi-objective Convex Programming Problems to Get the Best Compromise Solution

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ABSTRACT

A multi-objective convex programming problem has several convex objectives that need to be minimized or maximized over a convex set of constraints. When these objectives are conflicting with each other, the set of compromise solutions is obtained rather than one optimal solution. Accordingly, the best compromise solution is a very important topic for researchers because it considers the backbone of multi-objective optimization problems. This paper introduces two methods to get the best compromise efficient solution of multi-objective convex programming problems. The first suggested method develops Alia's method. For more flexibility, this first proposed method is combined with the distance based method. Through the search process of the best efficient solution, these two proposed methods produce the quality efficient set. Actually, their solution is valid for any number of objectives based on a new way that calculates the normal of objectives. Besides, some examples are presented to illustrate these methods, and the results are compared with other works. Interestingly, the results indicate effectiveness and robustness of the proposed methods in solving such problems. Also, it is concluded that more research is required on this topic.

Keywords: Multi-objective convex programming problems, Utopia point, Compromise solution, Alia's method, The distance based method

INTRODUCTION

It's for granted that the multi-objective optimization problem involves optimization of several objective functions simultaneously. These objective functions are measured in different units and they often are competing or conflicting in nature. Therefore, the decision-maker searches a compromising solution instead of finding an optimal solution for every objective function. There is a set of solutions that are equally good for such optimization of conflicting objectives problems. Indeed, these feasible solutions are Pareto-optimal, efficient, non-dominated, non-inferior, or compromise solutions. An efficient solution improvement is a movement from one feasible solution to another that can make:

- A feasible solution is strictly better than others in at least one objective.
- There is a feasible solution becoming better than others for all objectives.

Two spaces are considered in the multi-objective optimization problems: The n-dimensional space of the decision variables and the k-dimensional space of objective functions. **Figure (1***)* shows a two-dimensional representation of the mapping from the decision space into the objective space. For every solution in the decision space, there is a point on the objective space. As can be seen, feasible solutions on the bold curve of the decision space are called non-dominated solutions. But, the solutions on the red curve that are called Pareto-optimal in the objective space lie on the Pareto-optimal front. All Pareto-optimal solutions are non-dominated. Otherwise, a solution where an objective function can be improved without reducing the objective function of the other is called dominated solution or non-Pareto-optimal solution (Ashis *et al*., 2013; Jeffrey, 1992; Kaisa, 2004; Nyoman, 2018; Zhiyuan, and Gade, 2017).

The proposed methods hope to find the best feasible solution that has the shortest distance to the utopia point. The idea of paper comes from developing existing Alia's method based on a new treatment for the normal of objectives. The second work integrates the first proposed method with the distance-based method for better performance.

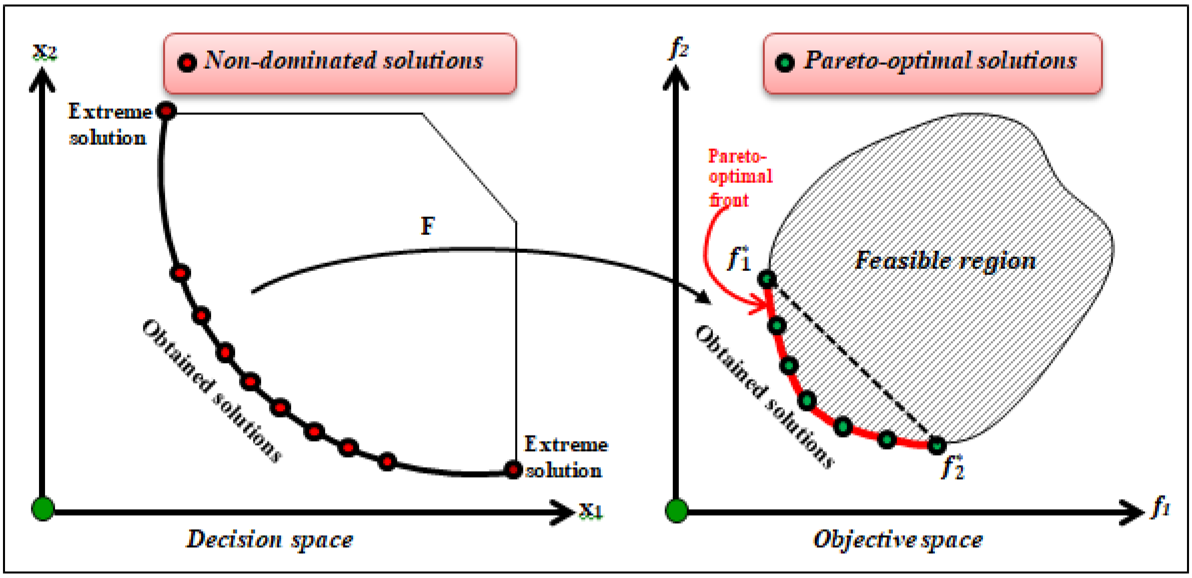

Figure 1: Mapping from a decision space onto an objective space for a multi-objective optimization problem, where both decision variables and objectives are to be minimized

This paper consists of the following sections: Section **2** presents the major fundamentals and definitions. Section **3** provides the proposed normal model that helps to get the best efficient solution. The proposed methods with their theorems, corollaries, properties, and advantages are introduced in section **4**. Section **5** gives some examples that illustrate the effectiveness of these methods and performs a comparison with the previous work. The final section offers conclusions and future perspectives.

2. THE MAJOR FUNDAMENTALS AND DEFINITIONS

In this section, some fundamentals and definitions related to the multi-objective convex programming problems are reviewed that will be used in the paper.

2.1. Problem Formulation

A mathematical model of the minimization multi-objective convex programming problem **(MOCPP)** can be stated as follows:

(MOCPP): Minimize $\mathbf{F(x)} = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x})),\ k \geq 2,$

Subject to $M = \{\mathbf{x} \in R^n / g_r(\mathbf{x}) \leq 0,\ r=1, 2, \ldots, m\}.$ **[1]**

Where:

$\mathbf{F(x)} = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x}))$ is a vector of k objective functions, and k is used to identify the number of objective functions,
$g_r(\mathbf{x})$ for all r = 1, 2, ... , m is a set of (inequality) constraints that is convex,
x is an n-vector of decision variables,

The set M is a non-empty and feasible region included in R^n that is determined by the constraints on the multi-objective convex problem.

Assume that

$\mathbf{f_i(x^*) = minimize\ f_i(x),\ i= 1, 2, \ldots, k,}$

Subject to $\mathbf{x \in M}$. **[2]**

2.2. Convexity

A function $\varphi(\mathbf{x})$: $\mathbf{S} \rightarrow R$ is called convex over a convex set **S**, if for any two points $x_1, x_2 \in \mathbf{S}$ and $\theta \in [0, 1]$, as in the **Figure (2)** below. Then:

$\varphi(\theta x_1 + (1-\theta) x_2) \leq \theta\, \varphi(x_1) + (1-\theta)\, \varphi(x_2), \quad 0 \leq \theta \leq 1.$ **[3]**

If the reverse inequality of the previous inequality holds, the function is concave. Thus $\boldsymbol{\varphi}(\mathbf{x})$ is concave if $-\boldsymbol{\varphi}(\mathbf{x})$ is convex. Linear functions are convex and concave at the same time.

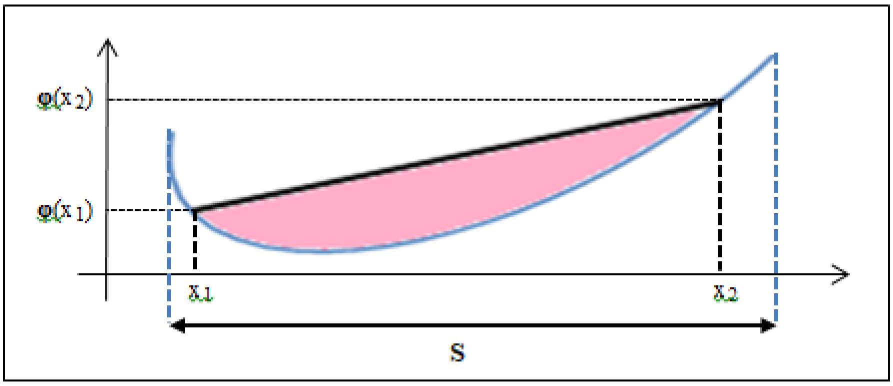

Figure 2: Illustration of a convex function

A set **S** ⊆ R^n is convex set if the line segment between any two points in **S** lies in **S**, i.e., if for every pair of points x_1, x_2 ∈ **S** and for every θ ∈ [0, 1], hence

x= $\theta x_1 + (1 - \theta) x_2 \in$ **S**, $0 \leq \theta \leq 1$. **[4]**

So, for instance, the sets shown in the top of **Figure (3)** are convex, but the other sets are not (David and Yinyu, 2008; Stephen and Lieven, 2009, Nisheeth, 2020).

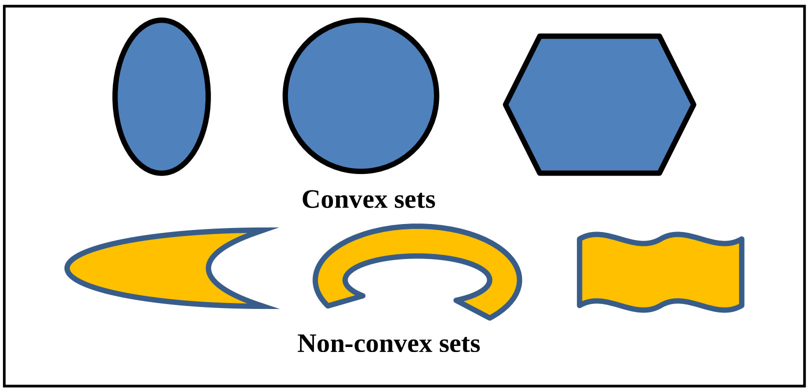

Figure 3: Some simple convex and non-convex sets

2.2.1. Convex Hull

The convex hull of a set **S** is the set of all convex combinations of points in **S**. Where, ***all convex combination***s of $x_1, \ldots, x_k$: any point **x** of the form **[4]** (Stephen and Lieven, 2009). The smallest convex set containing **S**. **Figure (4*)*** illustrates the definition of a convex hull.

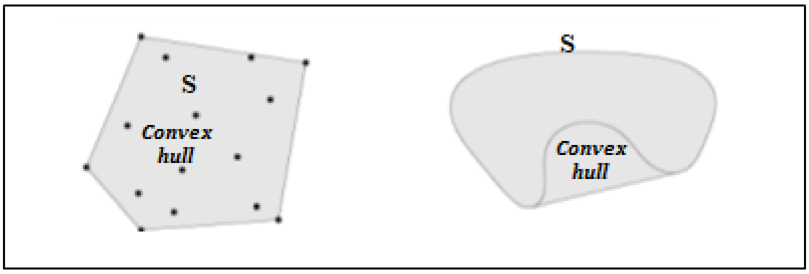

Figure 4: The convex hulls of two sets in R^2

2.2.2. Convex Programming

Convex programming studies the case when the objective function is convex (minimization) or concave (maximization) and the constraint set is convex. This can be viewed as a particular case of nonlinear programming or as a generalization of linear or convex quadratic programming.

- **Linear programming (LP)** is a type of convex programming in which the objective and all constraint functions are linear. Such a set is called a polyhedron or a polytope if it is bounded.
- ***Second-order cone programming*** **(SOCP)** is a convex program and includes certain types of quadratic programs.
- ***Semi-definite programming*** **(SDP)** is a subfield of convex optimization where a semi-definite constraint on matrix variables replaces the non-negativity constraint. It is a generalization of linear and convex quadratic programming.

- ***Conic programming*** **(CP)** is a general form of convex programming. **LP**, **SOCP** and **SDP** can all be viewed as conic programs with the appropriate type of cone.
- ***Geometric programming*** **(GP)** is a type of nonlinear optimization problem whose objective and constraints have a particular form. The decision variables must be strictly positive (non-zero, non-negative) quantities that can be transformed into a convex program.
- ***Quadratic programming*** **(QP)** is a type of convex programming that allows the objective function to have quadratic terms, while the feasible set is specified with equality and inequality linear constraints (Lieven and Stephen, 1996; Michael and Leslie, 2002; M. Upmanyu and R. R. Saxena, 2015; Murshid *et al*., 2018).

2.3. Efficient Solution

A decision vector $x^* \in S$ is said to be an ***efficient solution*** if there does not exist another decision vector $x \in S$ such that $f_i(x) \le f_i(x^*)$ for $i = 1, 2, \dots, k$ and $f_j(x) < f_j(x^*)$ for at least one index j.

2.4. Pareto-optimal Front

The collection of all Pareto-optimal solutions is called the Pareto-optimal set. The Pareto-optimal set image is referred to as Pareto-optimal front (efficient frontier, or tradeoff surface).

2.5. Utopia (Ideal) Point

The point $(f_1(x_1^*), f_2(x_2^*), \dots, f_k(x_k^*))$ in the objective space is called *utopia (ideal) point*.

2.6. Nadir Point

It is constructed from the worst objective values over the efficient set of a multi-objective optimization problem (Kalyanmoy, 2001).

2.7. Compromise Programming (CP)

Compromise programming is a complement to the multi-objective programming problem, and it allows reducing the set of efficient solutions to a more reasonable size without demanding any information of the decision-maker (DM). The graphical representation of a compromised surface is shown in ***Figure (5)*** for a bi-objective problem, which illustrates the objective function space and their feasible region with nadir point and utopia point. Throughout the compromise programming, the multi-objective problem is transformed into a single objective problem. Compromise programming assumes that any DM seeks a solution as close as possible to the ideal point (Daniel and Brian, 2010; Murshid *et al*., 2018; Ngo Tung *et al*., 2021; Stephen and Lieven, 2009).

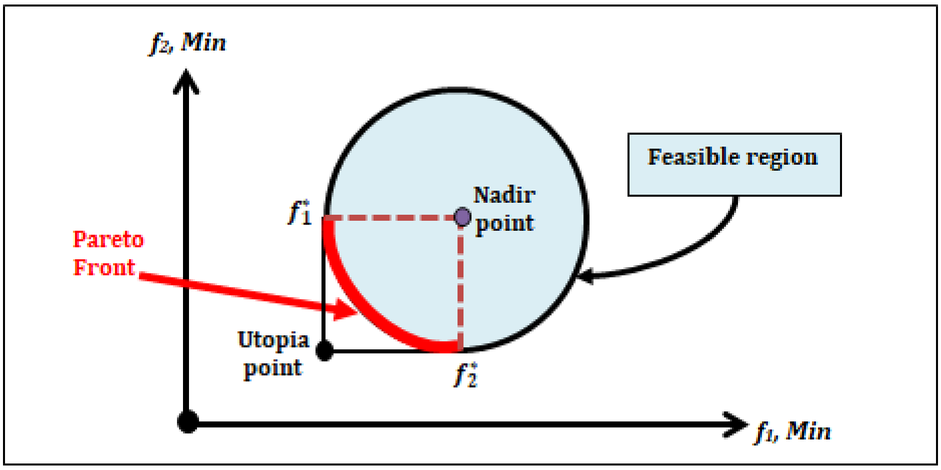

Figure 5: A representation of a compromise surface

2.8. Distance Function

The concept of distance is not used in its geometric sense but as a proxy measure for human preferences. The best compromise solution is the nearest solution to the ideal point. To measure the distance between two points A and B, various distance functions have been used in the literature, in which the Euclidean distance function is the most widely used one. The Euclidean distance between two points in a bi-dimensional space is the square root of the sum of the squared differences between each point's first and second components. It can be graphically represented by the straight line between two points (Janett and Yan 2010; Murshid *et al*., 2018; Stephen and Lieven, 2009; Li *et al*., 2016). Extension to the k-dimension space, let $A = (a_1, a_2, \dots, a_k)$ and $B = (b_1, b_2, \dots, b_k)$, the Euclidean distance is used as follows:

Distance (A, B) = $\sqrt{\sum_{j=1}^{k}(a_j - b_j)^2}$. **[5]**

2.9. Best Compromise Solution

The best compromise solution on the efficient front is a feasible solution with the shortest distance to the utopia solution. Also, it is the point of common adjacent between the efficient front and a utopia hypersphere (Alia, 2016; Stephen and Lieven, 2009). The following **Figure (6)** shows a representation of compromise utopia distances in a bi-objective problem:

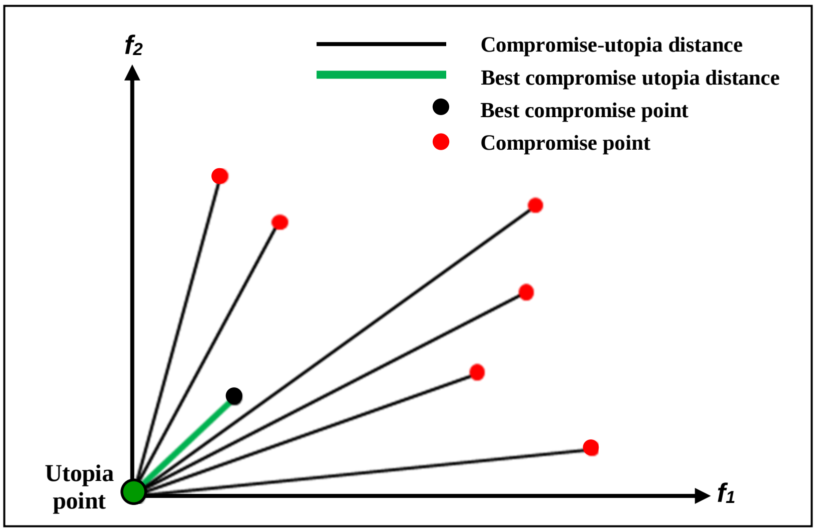

Figure 6: A representation of compromise utopia distances in a bi-objective problem

2.10. Alia's Method

Alia's method solves the multi-objective convex programming problems for all weights when the objectives are conflicting. This method combines the main positive features of the hybrid method and the normal boundary intersection (NBI) method. It contains a group of steps to obtain Alia point. Alia point is the best point, or it could be very close to the best point based on its distance from the utopia point.

The following problem denoted as (AP).

(AP): Minimize $\left(\sum_{i=1}^{k} w_i f_i(\mathbf{x}) + \| N \|^2 \delta\right)$,

Subject to $f_i(x) - n_i\,\delta \le f_i^*$, i= 1, 2, 3, ... , k,
$x \in M$. **[6]**

Where:

$\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ is a vector of the decision variables, n is a number of the decision variables.
$w_1, w_2, \dots, w_k$ are weights of the objective functions $f_i(x)$, $w_i > 0$, i= 1, 2, ... , k, $\sum_{i=1}^{k} w_i = 1$.
k is a number of objective functions.
$f_i^* = \min f_i$, is an optimal objective value for every objective (f_i) of the problem over M.
$N\delta$ is the normalized controlling vector.
$N = (n_1, n_2, \dots, n_k)^T$ is the normal vector directed in the positive direction to the utopia hyperplane.
δ (variable) is clearly positive due to the feasibility of the constraints.

Lemma 1:

*Let the constraints of Alia's problem (AP) satisfy the Slater constraint qualification (or any other constraint qualifications). If for $\overline{w} > 0$, $(\overline{x}, \overline{\delta})$ is an optimal solution of (AP), then $\overline{x}$ is an efficient solution of **(MOCPP)**.*

Theorem 1:

*If for $\overline{w} > 0$, $(\overline{x}, \overline{\delta})$ is an optimal solution of (AP) such that $f_i(\overline{x}) - n_i\,\overline{\delta} = f_i^*$, i= 1, 2, 3, ... , k, then $\overline{x}$ will be an Alia efficient point for **(MOCPP)*** (Alia, 2016).

2.11. The Distance Based Method

The distance-based method is used to decide the best compromise solution for multi-objective linear programming problems. It can obtain the compromise solution without any preference and for different preferences. Its applications are discussed for transportation and assignment problems involving multiple and conflicting objectives. The multi-objective problem is simply reduced to the following single objective problem:

(DM): Minimize $\left(\sum_{i=1}^{k} |f_i - f_i^{Ideal}|\,(1 - w_i)\, d\right)$,

Subject to

$f_i \leq f_i^{Ideal} + d\,(1 - w_i)$, $i = 1, 2, 3, \dots, k$,

$\mathbf{x} \in M$, $\mathbf{x} \geq 0$. **[7]**

Where:

$w_i > 0$, $i = 1, 2, \dots, k$, $\sum_{i=1}^{k} w_i = 1$, k is a number of objective functions, **[8]**

f_i^{Ideal} is an optimum value of *i*th objective obtained as a single objective problem or the ideal value of the *i*th objective,

$\mathbf{x}$ is a decision vector, and

d is a general deviational variable.

Lemma 2:

The smallest face containing all edges incident to a common vertex of a pointed polyhedron is all polyhedron (Murshid *et al.*, 2018).

3. THE PROPOSED NORMAL MODEL

In order to help the decision makers to find the best efficient solution of the multi-objective convex programming problems, a new "normal" namely, **Alia's normal model (ANM)** is proposed. Its formulation is as follows:

(ANM): Minimize $\left(\sum_{i=1}^{k} n_i\right)$,

Subject to $p\,\mathbf{t} - \sum_{i=1}^{k} n_i \leq f_i^*$, $i = 1, 2, 3, \dots, k$, **[9]**

Where: p is the payoff matrix, and

$\mathbf{t} = (t_1, t_2, \dots, t_k)^{\mathbf{T}}$ is a decision vector of the proposed normal model.

The resulting normal has the minimum value corresponding to the individual optimal of objectives. It is called Alia normal or **A.** normal.

Theorem 2:

*If the normal of objectives **(**$N = (n_1, n_2, \dots, n_k)^T$**)** is generated from Alia's normal model **(ANM)**, then it has the optimal solution.*

Proof:

The proof of this theorem is simple consequence of linear programming optimization theory. It can be given as follows: Based on **Lemma 2**, and let $N = (n_1, n_2, \dots, n_k)^{\mathbf{T}}$ be a vector of variables that optimizes (either minimizes or maximizes) the normal of objectives directed in the positive direction to the utopia hyperplane. This means that the vector $N \in R^k$ has the optimal solution of **(ANM)** such that: $N^* =$ optimal $(n_1, n_2, \dots, n_k)^{\mathbf{T}}$. Since the left hand side of constraints is the payoff matrix of the multi-objective convex programming problem, and the right hand side of constraints is the ideal values. By applying the linear programming optimization theory to **(ANM)**, the proof is obtained.

4. THE PROPOSED METHODS TO GET THE BEST COMPROMISE SOLUTION

There are two proposed methods to provide the best compromise efficient solution of multi-objective convex programming problems. These are extremely valuable for decision-makers.

4.1. The First Proposed Method (AAM)

This section develops the existing Alia's method based on the new normal of objectives. It is called **the advanced Alia's method (AAM)**. This method gives the best compromise solution for all or some weights of objectives.

In the following, a formal characterization of the best compromise solution of **(MOCPP)** will be given. The ***convex hull of individual minima*** in the objective space **(CHIM)** is denoted by ***H***. Where, ***H*** is the image of constraints ***(C)*** in the decision space that is mapped by the vector function ***F*** in the objective space. An element $\boldsymbol{P} \in \boldsymbol{H}$ is a vector $(P_1, \dots, P_{\mathbf{H}})^{\boldsymbol{T}}$, where P_i is the *i*th objective function value. First of all, the concept of the best compromise solution is defined as follows:

Let ***P*** be a Pareto optimal solution for **(MOCPP)**; ***P*** is called the best compromise solution if P_i reaches the minimum distance value for $i = 1, \dots, \boldsymbol{H}$.

Theorem 3:

A solution $\mathbf{x}^* \in M$ is an efficient solution for **(MOCPP)** if and only if $\mathbf{x}^*$ is an optimal solution of **(AAM)**.

Proof:

This proof follows directly from the definition of Alia point. Let for $\overline{w} > 0$, $(\overline{x}, \overline{n}_i, \overline{\delta})$ be an optimal solution for **(AAM)**. Where, **(AAM)** uses Alia's normal model (that provides the optimal values for $\overline{n}_i$, $i = 1, 2, 3, \dots, k$) in **(AP)**. Since Alia point is an optimal solution for Alia's problem, and at the same time it is an efficient solution for **(MOCPP)**, then from **Lemma 1** and **theorem 1** $\overline{x}$ is the ***efficient solution*** for **(MOCPP)**. Thus the result holds.

Theorem 4:
An optimal solution $\mathbf{x}^*$ for **(AAM)** is said to be the best compromise solution (**P**) (in the objective space $\mathbf{E^p}$ of **(MOCPP)**) in $\mathbf{H} \subseteq \mathbf{F(C)}$ if and only if there does not exist another solution ($\mathbf{y}$) $\in$ **F(C)** such that:
Distance (y) > distance (x*). This means that $\mathbf{x}^*$ has minimum distance from the utopia point.

Proof:
Utilizing Kuhn- Tucker conditions and choosing Alia normal at optimal values to solve **(AAM)**. Whereas, $\mathbf{x}$, n_i, δ are selected at optimal values for this problem. That leads to the difference between $f_i(\mathbf{x})$ and f_i^* is at minimum value. Then, the proof is provided.

4.2 . The second proposed method (MAM)

The second proposed method can be formulated on the basis of integrating the developing Alia's method and the distance based method. It is called **the mixed Alia's method (MAM)**. This method consists of the following steps:

1. Calculating the individual minima (f_i^*) of the performances f_i with i= 1, 2, … , k, which are determined from solving **(MOCPP)** for all individual objective (f_i).
2. Constructing the payoff matrix (p).
3. Determining **Alia's normal calculation** (N) of objective vectors by **(ANM)**.
4. Formulating the multi-objective convex programming problem as a single objective optimization problem using the proposed method. Solving this formulation by any of the available solvers such as LINGO (that solves the nonlinear optimization programming problems) to obtain an efficient solution.
5. Calculating the distance of the resulting efficient point. If this efficient point has the minimum distance, thus it is obtained and the process is terminated, otherwise go to the next step.
6. Suggesting new weights (and sometimes new value (s) of normal) for the objectives and repeating from Step **4** to Step 6 until the best efficient solution is obtained.

Now, the problem to be solved is

(MAM): Minimize $\left(\sum_{i=1}^{k} w_i\, d \,|\, f_i(\mathbf{x}) - f_i^* | + \| N \|^2\, \delta\right)$,
Subject to
$f_i(\mathbf{x}) - n_i\,\delta - \sum_{i=1}^{k} w_i\, d \le f_i^*$, i= 1, 2, 3, ... , k,
$\mathbf{x} \in M = \{\mathbf{x} \in R^n \,/\, g_r(\mathbf{x}) \le 0, \quad r = 1, 2, \dots, m\}$. **[10]**

Where:
$\mathbf{x} = (x_1, x_2, \dots, x_n)^{\mathbf{T}}$ is a vector of the decision variables, n is a number of the decision variables.
$w_1, w_2, \dots, w_k$ are weights of the objective functions $f_i(x)$, $w_i \ge 0$, i= 1, 2, ... , k, $\sum_{i=1}^{k} w_i = 1$.
d is a general deviational variable for all objectives.
k is a number of objective functions.
f_i^* is an optimal objective value for every objective "i" of the problem over M.
$N\delta$ is the normalized controlling vector.
$N = (n_1, n_2, \dots, n_k)^{\mathbf{T}}$ is the normal vector directed in the positive direction to the utopia hyperplane.
δ (variable) is clearly positive due to the feasibility of the constraints.

Theorem 5:
*A solution $\mathbf{x}^* \in M$ is an efficient solution for **(MOCPP)** if and only if $\mathbf{x}^*$ is an optimal solution to **(MAM)**.*

Proof:
The mixed Alia's method consists of **(AAM)** and **(DM)**. Every problem has optimal solution. Each optimal solution is also considered an efficient solution for **(MOCPP)**. Therefore, it could be said that $x^* \in$ M of mixed Alia's problem is an efficient solution to **(MOCPP)**, thus the result holds.

Theorem 6:
An optimal solution $\mathbf{x}^*$ of **(MAM)** is said to be the best compromise solution (**P**); in the objective space $\mathbf{E^p}$ of **(MOCPP)**; in $\mathbf{H} \subseteq \mathbf{F(C)}$, if and only if there does not exist another solution ($\mathbf{y}$) $\in$ **F(C)** such that:
Distance (y) > distance (x*).

Proof:
In order to prove this theorem, let Kuhn-Tucker conditions (Mokhtar *et al.*, 2006) are formulated for the proposed problem as follows:

$$\sum_{i=1}^{k} w_i\, d\, \frac{\partial\, |f_i(\mathbf{x}) - f_i^*|}{\partial x_j} + \sum_{i=1}^{k} \mu_i \frac{\partial f_i(\mathbf{x})}{\partial x_j} + \sum_{r=1}^{m} \alpha_r \frac{\partial g_r(\mathbf{x})}{\partial x_j} = 0, \quad j = 1, 2, 3, \dots, n, \qquad [11]$$

$$\sum_{i=1}^{k} w_i\, |f_i(\mathbf{x}) - f_i^*| = \mu_i \sum_{i=1}^{k} w_i \qquad [12]$$

$\sum_{i=1}^{k} n_i\ \mu_i = \| N \|^2,$ **[13]**

$f_i(\mathbf{x}) - n_i\ \delta - \sum_{i=1}^{k} w_i\ d \le f_i^*,$ i= 1, 2, 3, ... , k, **[14]**

$g_r(\mathbf{x}) \le 0,$ r = 1, 2, ... , m, **[15]**

$\mu_i\ (f_i(\mathbf{x}) - n_i\ \delta - \sum_{i=1}^{k} w_i\ d - f_i^*) = 0,$ i= 1, 2, 3, ... , k, **[16]**

$\alpha_r\ g_r(\mathbf{x}) = 0,$ r = 1, 2, ... , m, **[17]**

$\mu_i \ge 0,$ i= 1, 2, 3, ... , k, **[18]**

$\alpha_r \ge 0,$ r = 1, 2, ... , m, **[19]**

$\eta_i \ge 0,$ i= 1, 2, 3, ... , k, **[20]**

Since $\sum_{i=1}^{k} w_i = 1$, and w_i, $\mu_i \ge 0$, it can be easily verified that $\sum_{i=1}^{k} w_i\ | f_i(\mathbf{x}) - f_i^*| = 0$ in equation **[12]**. But from equation **[16]**, it is clear that: $f_i(\mathbf{x}) - f_i^* = n_i\ \delta + \sum_{i=1}^{k} w_i\ d$. Whereas, $\mathbf{x}$, n_i, δ, d are selected at optimal values for this problem. Also, it can be seen from equation **[13]** that: $| f_i(\mathbf{x}) - f_i^*| = \| N \|$. That leads to the difference between $f_i(\mathbf{x})$ and f_i^* be at minimum value. Thus, this model aims at obtaining the solution with the shortest distance from utopia point. With this way, the theorem is proved.

Thereafter, the following corollaries are constructed.

Corollary 1:
The best compromise solution by the proposed methods (**(AAM)** and **(MAM)**) of the primal **(MOCPP)** (that satisfies Slater's condition) is also the best compromise solution of its dual, and conversely.

Proof:
Since ***(MOCPP)*** satisfies Slater's condition (indeed, strong duality always holds for this convex problem (*The definition is mentioned by Stephen and Lieven (2009)*)). Then, it satisfies the optimality conditions (or Kuhn- Tucker conditions). Consequently, the optimum value of the primal ***(MOCPP)* =** P^* is equal to the optimum value of its dual (D^*). Since the set of Pareto-optimal solutions of **(MOCPP)** corresponding to both problems primal and dual is the same. Then, the best compromise solution by the proposed methods (**(AAM)** and **(MAM)**) is the same of such Pareto-optimal set; that has the minimum Euclidean distance of solution from utopia point. Thus the result holds.

Corollary 2:
In the objective space $\mathbf{E}^{\mathbf{p}}$, it is unnecessary for the proposed normal for objectives to go through the utopia point to get the best compromise solution of **(MOCPP)**.

Proof:
From the definition of Alia point (Alia, 2016), it must pass the problem's normal and the convex hull of individual minima **(CHIM)** from the utopia point. Alia point is the best point or so close to it in efficient front. Then, in order to get the best compromise solution, it is not necessary for Alia normal to pass by the utopia point. The corollary is completely proved.

Corollary 3:
If all individual minima (or maxima) f_i^* of the performances f_i with i= 1, 2, … , k for **(MOCPP)**, are zeros, this means that all optimal values of n_i are zeros from solving **(ANM)**.

Proof:
The proof is given based on linear programming optimization theory.
In the following, ***Figure (7)*** illustrates a graphical representation of the proposed methods in a bi-objective problem:

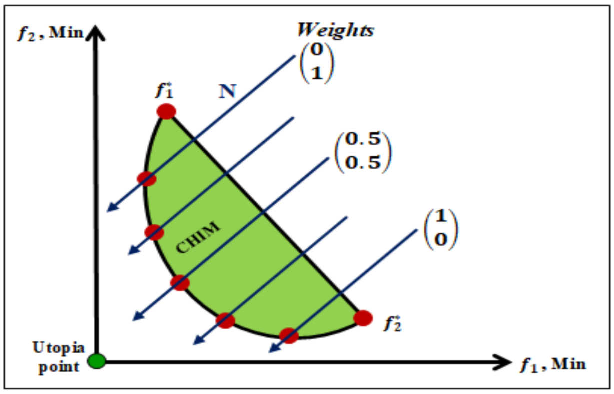

Figure 7: A graphical representation of the proposed methods for a bi- objective problem

4.3. Some important notes:

1. **Alia's normal model** has at least one zero value because the vector N is perpendicular on the **CHIM**. So, when the value of n_i is zero in the second proposed method **(MAM)**, it can be represented as n_i = zero or any real number to get the best solution.
2. If all values of n_i are zeros, the solution of advanced Alia's method **(AAM)** is infeasible. In this case, any other real numbers are chosen for n_i.
3. When f_i^* has real value, its n_i is minimized in Alia's model of normal. This means that when the value of f_i^* is zero, n_i can be ignored in minimizing the objective function of its normal formulation.
4. If there are conflicting and non-conflicting objectives, the normal non-conflicting objectives in Alia's model can be optimized.
5. The proposed methods can be implemented easily by starting with equality weights ($\sum_{i=1}^{k} w_i / k$). Then, these values can be increased and/ or decreased until the best compromise solution is obtained.
6. In the second proposed method, there are two types of weights and every type is equal to one. The first type is used for scalarizing the multi-objective problem, which is called the first group of weights. But, the second type of weights is included in constraints of **(MAM)**; which is called the second group of weights.

The flowchart of the proposed methods can be given in ***Figure (8)***. As can be seen in this Figure, the optimization process continues with the computation until the best compromise solution is obtained.

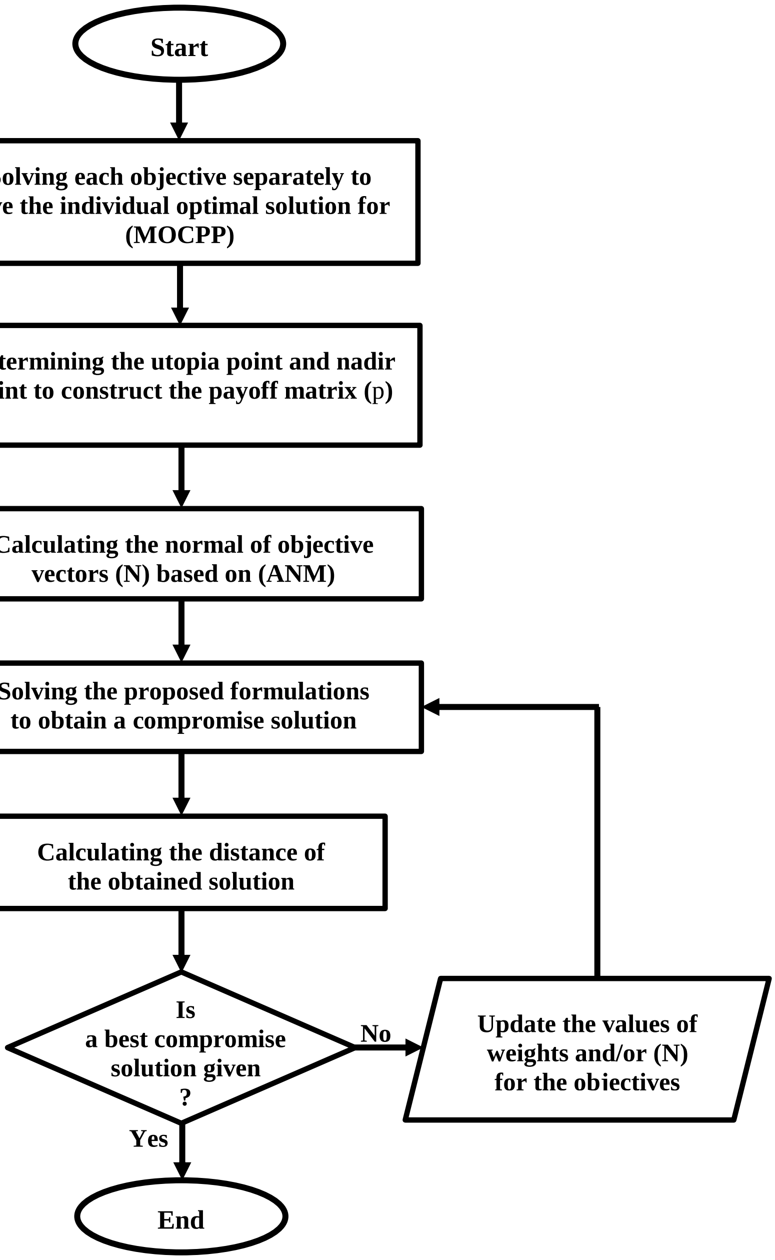

Figure 8: Flowchart representation of the proposed methods

4.4. The properties of the proposed methods:

1. The solution of the proposed methods (**(AAM)** and **(MAM)**) is optimal and also it is considered the efficient solution for the given multi-objective convex programming problem **(MOCPP)**.
2. In **(MAM)**, if the number of objectives is more than two objectives, the values of the first group of weights may be different from the second group of weights. But, the total of weights " $\sum_{i=1}^{k} w_i$" for each group is equal to one.
3. **(MAM)** solves **(MOCPP)** with any nonlinear solver even if **(MOCPP)** is linear.
4. The new normal formulation of objective vectors is better than the problem's normal to get the best compromise solution. In addition, it gives flexibility to select the values of n_i that can achieve the minimum distance from the utopia point.
5. If there is more than one zero value of n_i for **(ANM)** optimization, it may be concluded that the value of individual optimal of objective (s) is zero, or the inequality constraint (s) is tight.

4.5. The advantages of the proposed methods:

1. Generally, the proposed methods (**(AAM)** and **(MAM)**) provide an infinite number of robust efficient solutions to their problems which are very much useful to decision maker for selecting the best solution.
2. The best compromise solution is found in the efficient set of multi-objective convex programming problems, whether these objectives are conflicting or not conflicting.
3. Calculating the normal of objectives (N) by Alia's model helps to find the best solution easily with saving time.

5. NUMERICAL EXAMPLES

Some numerical examples from previous work demonstrate the potentiality of the presented methods.

Example (1):
This example from work by Murshid *et al*. (2018) is given as follows:

Minimize: F= {($f_1 = x_{11}+ 2x_{12}+ 7x_{13}+ 7x_{14}+ x_{21}+ 9x_{22}+ 3x_{23}+ 4x_{24}+ 8x_{31}+ 9x_{32}+ 4x_{33}+ 6x_{34}$**),**
($f_2 = 4x_{11}+ 4x_{12}+ 3x_{13}+ 3x_{14}+ 5x_{21}+ 8x_{22}+ 9x_{23}+ 10x_{24}+ 6x_{31}+ 2x_{32}+ 5x_{33}+ x_{34}$**)},**
Subject to:
$x_{11}+ x_{12}+ x_{13}+ x_{14}= 8,$
$x_{21}+ x_{22}+ x_{23}+ x_{24}= 19,$
$x_{31}+ x_{32}+ x_{33}+ x_{34}= 17,$
$x_{11}+ x_{21}+ x_{31}= 11,$
$x_{12}+ x_{22}+ x_{32}= 3,$
$x_{13}+ x_{23}+ x_{33}= 14,$
$x_{14}+ x_{24}+ x_{34}= 16,$
$x_{ij} \geq 0,\ i= 1, 2, 3, \text{ and } j= 1, 2, 3, 4.$
By solving each objective separately, the payoff matrix is: $\begin{pmatrix} f^*_{11} & f^*_{12} \\ f^*_{21} & f^*_{22} \end{pmatrix} = \begin{pmatrix} 143 & 208 \\ 265 & 167 \end{pmatrix}$, the problem's normal is (98, 65). The mathematical programming of **(MAM)** for the above problem is written as follows:

Minimize: $\mathbf{0.56\ d}\ |\ (x_{11}+ 2x_{12}+ 7x_{13}+ 7x_{14}+ x_{21}+ 9x_{22}+ 3x_{23}+ 4x_{24}+ 8x_{31}+ 9x_{32}+ 4x_{33}+ 6x_{34} - \mathbf{143})\ | +$
$\mathbf{0.44\ d}\ |\ (4x_{11}+ 4x_{12}+ 3x_{13}+ 3x_{14}+ 5x_{21}+ 8x_{22}+ 9x_{23}+ 10x_{24}+ 6x_{31}+ 2x_{32}+ 5x_{33}+ x_{34} - \mathbf{167})\ | +$
$13829\ \delta,$
Subject to:
$x_{11}+ 2x_{12}+ 7x_{13}+ 7x_{14}+ x_{21}+ 9x_{22}+ 3x_{23}+ 4x_{24}+ 8x_{31}+ 9x_{32}+ 4x_{33}+ 6x_{34} - \mathbf{98}\ \delta - \mathbf{0.56\ d} \leq 143,$
$4x_{11}+ 4x_{12} + 3x_{13} + 3x_{14}+ 5x_{21}+ 8x_{22}+ 9x_{23}+ 10x_{24}+ 6x_{31}+ 2x_{32}+ 5x_{33}+ x_{34} - \mathbf{65}\ \delta - \mathbf{0.44\ d} \leq 167,$
$x_{11}+ x_{12}+ x_{13}+ x_{14}= 8,$
$x_{21}+ x_{22}+ x_{23}+ x_{24}= 19,$
$x_{31}+ x_{32}+ x_{33}+ x_{34}= 17,$
$x_{11}+ x_{21}+ x_{31}= 11,$
$x_{12}+ x_{22}+ x_{32}= 3,$
$x_{13}+ x_{23}+ x_{33}= 14,$
$x_{14}+ x_{24}+ x_{34}= 16,$
$x_{ij} \geq 0,\ i= 1, 2, 3, \text{ and } j= 1, 2, 3, 4.$

The steps to calculate the best compromise solution using the proposed methods for this problem are given as follows:
i) Finding Alia normal through the following model:

Minimize: $(n_1+ n_2)$,
Subject to:
$143\ t_1+ 208\ t_2 - n_1 \leq 143,$

$265\ t_1 + 167\ t_2 - n_2 \leq 167$.
The resulting solution is $n_1 = n_2 = 0$.
ii) The mathematical programming models **((AAM)** and (**MAM**)) are written in the following **TABLE 1**.

Table 1: The Two Proposed Methods for Example (1)

The first method (AAM)	The second method (MAM)
Minimize: **0.5** $(x_{11} + 2x_{12} + 7x_{13} + 7x_{14} + x_{21} + 9x_{22} + 3x_{23} + 4x_{24} + 8x_{31} + 9x_{32} + 4x_{33} + 6x_{34})$ + **0.5** $(4x_{11} + 4x_{12} + 3x_{13} + 3x_{14} + 5x_{21} + 8x_{22} + 9x_{23} + 10x_{24} + 6x_{31} + 2x_{32} + 5x_{33} + x_{34})$ + **0.50629442** δ,	**Minimize:** **0.5561 d** $\mid (x_{11} + 2x_{12} + 7x_{13} + 7x_{14} + x_{21} + 9x_{22} + 3x_{23} + 4x_{24} + 8x_{31} + 9x_{32} + 4x_{33} + 6x_{34} - \mathbf{143}) \mid$ + **0.4439 d** $\mid (4x_{11} + 4x_{12} + 3x_{13} + 3x_{14} + 5x_{21} + 8x_{22} + 9x_{23} + 10x_{24} + 6x_{31} + 2x_{32} + 5x_{33} + x_{34} - \mathbf{167}) \mid$ + **0**(δ),
Subject to:	**Subject to:**
$x_{11} + 2x_{12} + 7x_{13} + 7x_{14} + x_{21} + 9x_{22} + 3x_{23} + 4x_{24} + 8x_{31} + 9x_{32} + 4x_{33} + 6x_{34} - \mathbf{0.5561}\ \delta \leq 143$,	$x_{11} + 2x_{12} + 7x_{13} + 7x_{14} + x_{21} + 9x_{22} + 3x_{23} + 4x_{24} + 8x_{31} + 9x_{32} + 4x_{33} + 6x_{34} - \mathbf{0}\ (\delta) - \mathbf{0.5561\ d} \leq 143$,
$4x_{11} + 4x_{12} + 3x_{13} + 3x_{14} + 5x_{21} + 8x_{22} + 9x_{23} + 10x_{24} + 6x_{31} + 2x_{32} + 5x_{33} + x_{34} - \mathbf{0.4439}\ \delta \leq 167$,	$4x_{11} + 4x_{12} + 3x_{13} + 3x_{14} + 5x_{21} + 8x_{22} + 9x_{23} + 10x_{24} + 6x_{31} + 2x_{32} + 5x_{33} + x_{34} - \mathbf{0}\ (\delta) - \mathbf{0.4439\ d} \leq 167$,
$x_{11} + x_{12} + x_{13} + x_{14} = 8$,	$x_{11} + x_{12} + x_{13} + x_{14} = 8$,
$x_{21} + x_{22} + x_{23} + x_{24} = 19$,	$x_{21} + x_{22} + x_{23} + x_{24} = 19$,
$x_{31} + x_{32} + x_{33} + x_{34} = 17$,	$x_{31} + x_{32} + x_{33} + x_{34} = 17$,
$x_{11} + x_{21} + x_{31} = 11$,	$x_{11} + x_{21} + x_{31} = 11$,
$x_{12} + x_{22} + x_{32} = 3$,	$x_{12} + x_{22} + x_{32} = 3$,
$x_{13} + x_{23} + x_{33} = 14$,	$x_{13} + x_{23} + x_{33} = 14$,
$x_{14} + x_{24} + x_{34} = 16$,	$x_{14} + x_{24} + x_{34} = 16$,
$x_{ij} \geq 0$, i=1, 2, 3 and j=1, 2, 3, 4.	$x_{ij} \geq 0$, i=1, 2, 3 and j=1, 2, 3, 4.

The best compromise solution obtained by the distance based method (Murshid *et al*. 2018) is $x^*_{11} = 2$, $x^*_{12} = 3$, $x^*_{13} = 3$, $x^*_{14} = 0$, $x^*_{21} = 9$, $x^*_{22} = 0$, $x^*_{23} = 10$, $x^*_{24} = x^*_{31} = x^*_{32} = 0$, $x^*_{33} = 1$, $x^*_{34} = 16$, with $f^*_1 = 168$, $f^*_2 = 185$, and its distance is 30.806. But, the best compromise solution of the advanced Alia's method and the mixed method (either by the problem's normal or **A.** normal) provided by LINGO software is as follows: $x^*_{11} = 2.238735$, $x^*_{12} = 3$, $x^*_{13} = 2.761265$, $x^*_{14} = 0$, $x^*_{21} = 8.761265$, $x^*_{22} = 0$, $x^*_{23} = 10.23873$, $x^*_{24} = x^*_{31} = x^*_{32} = 0$, $x^*_{33} = 1$, $x^*_{34} = 16$ with $f^*_1 = 167.045$, $f^*_2 = 186.1936$. The minimum distance is 30.766. The weights of (**AAM**) and the first group of weights for **(MAM)** are **(**(0.2, 0.8), (0.3, 0.7), (0.4, 0.6), (0.5, 0.5), (0.6, 0.4), (0.7, 0.3), (0.8, 0.2), (0.9, 0.1)**)**. The second group of weights for **(MAM)** are $w_1 = 0.5561$ and $w_2 = 0.4439$. The selected values of **A.** normal for (**AAM**) are $n_1 = 0.5561$ and $n_2 = 0.4439$. But, **A.** normal for (**MAM**) is $n_1 = n_2 = 0$. The resulting solution of **(MAM)** using both normal models is the same. Consequently, these proposed models gave the accurate best compromise solution than other.

Example (2):
This example is taken from paper by P. K. De and Bharti (2011), where the linear multi-objective problem has the following form:
Minimize: F= {$(x_1 - 2x_2)$, $(-2x_1 - x_2)$**}**,
Subject to:
$-x_1 + 3x_2 \leq 21$,
$x_1 + 3x_2 \leq 27$,
$4x_1 + 3x_2 \leq 45$,
$3x_1 + x_2 \leq 30$,
$x_1, x_2 \geq 0$.
On solving all individual objective functions, optimal solutions can be obtained as: $f^*_1(x^*_1 = 0,\ x^*_2 = 7) = -14$ and $f^*_2(x^*_1 = 9, x^*_2 = 3) = -21$. The payoff matrix can be extracted as: $\begin{pmatrix} -14 & -7 \\ 3 & -21 \end{pmatrix}$.

Using LINGO software, the resulting solution is obtained as $x^*_1 = 4.8$, $x^*_2 = 7.4$, $f^*_1 = -10$, $f^*_2 = -17$, $C_1 = 17.4$, $C_2 = 27$, $C_3 = 41.4$, $C_4 = 21.8$, and its distance = 5.65685. In this example, both methods (**(AAM)** and **(MAM)**) give the best compromise solution using any value of $n_1 = n_2 > 0$ for all weights. The results of these models are more accurate than the solution by P. K. De and Bharti (2011) (that is: $x^*_1 = 6$, $x^*_2 = 7$ with distance = 6.32456).

Example (3):
Consider the linear multi-objective problem:
Minimize: F= {(x), (y)**}**,
Subject to:
$2x + y \geq 4$,
$2x + 3y \geq 8$,
$x + y \leq 4$,
$x \geq 0, y \geq 0$.

The normal of this problem is (1, 1) that gives the best solution = (1.6, 1.6) with distance = 2.2627 for all weights. Since the resulting normal of **(ANM)** is (0, 0), the proposed methods (**(AAM)** and **(MAM)**) use their normal as (0.1, 0.15) and the weights as (0.45, 0.55). The best compromise solution is x^*= 1.230769, y^*= 1.846154 with distance = 2.2188.
It is clear that the new normal gives better results than other normal of the problem. Alia's method can produce the best compromise solution using the new normal model for some weights. On the other hand, the mixed method uses **A.** normal = (0, 0), unlike the result of first method that gives infeasible solution in this case. Furthermore, the mixed method is more flexibility than the first method because it obtains the best solution for all weights (the first and the second group of weights ($w_i > 0$, $\sum_{i=1}^{k} w_i = 1$)) when the selected values of normal are n_1= 0.1, and n_2= 0.15 only.

Example (4):
Consider the convex four multiple criteria example that is taken from paper by Alia (2016):
Minimize: F= {(x), (y), (–x – 3y), (2x² – 4y)**}**,
Subject to:
$x + y \geq 2$,
$-x + y \leq 2$,
$3x + y \leq 6$,
$x \geq 0$, $y \geq 0$.
The utopia objective values are obtained by solving each objective separately as follows:
f_1^*= 0 attained at the point (0, 2),
f_2^*= 0 attained at the point (2, 0),
f_3^*= –10 attained at the point (1, 3), and
f_4^*= –10 attained at the point (1, 3).
It is clear that f_3, f_4 are non-conflicting, thus one of them can be deleted from the problem. Also, all objectives can be solved together.

The first three objectives (f_1, f_2, f_3):
Hence f_1^*= f_2^* = 0, n_3 is only minimized. In this case, new normal is (n_1= 1, n_2= 3, n_3= 0). But, the used normal is n_1= 1, n_2= 3, n_3= 1. Two groups of weights are used for the second proposed method: the first group has equal values. The second group is (w_1= w_2= 0, w_3= 1). The best compromise solution using the LINGO software is given by $\mathbf{x}^*$= 0.9090909, $\mathbf{y}^*$= 2.727273. It has distance = 3.015 from the utopia point (0, 0, –10). This solution is obtained by the first proposed method for all weights.

The three objectives (f_1, f_2, f_4):
The values of normal for **(AAM)** are n_1= 1, n_2= 3, n_3= 1.7901845, but the values used for **(MAM)** are n_1= n_3= 1, n_2= 3. The weights of objectives = (w_1= 0.1, w_2= 0.2117, w_3= 0.6883) are the same for two models. The second group of weights is (w_1= 0.0, w_2= 0.1117263, w_3= 0.8882737). The resulting solution is ($\mathbf{x}^*$= 0.3106, $\mathbf{y}^*$= 2.3106) with the distance from the utopia point (0, 0, –10) is 2.5177.

The four objectives (f_1, f_2, f_3, f_4):
When n_3, and n_4 are minimized by the proposed model of normal, the optimal values are (n_1= 1, n_2= 3, n_3= n_4= 0). The selected values of normal for **(AAM)** are: (n_1= 1, n_2= 3, n_3= n_4= 0.93865851). The weights assigned to objectives are (w_1= 0.13 or 0.10, w_2= 0.05 or 0.07, w_3= 0.04 or 0.01, w_4= 0.78 or 0.82). But, the values of normal for **(MAM)** are: (n_1= 1, n_2= 3, n_3= n_4 = 0.3527264). Its first group of weights is (w_1= 0.01, w_2= 0.02, w_3= 0.0756029, w_4= 0.8943971), and the second group of weights is (w_1= 0.01, w_2= 0.11, w_3= 0.1118723514, w_4= 0.7681276486). The best compromise solution for all objectives is **(**$\mathbf{x}^*$= 0.78235955 or 0.78235954, $\mathbf{y}^*$= 2.78235955 or 2.78235954**)** and its distance from the utopia point (0, 0, –10, –10) is 3.02. Both proposed methods gave their solution for some weights.

6. CONCLUSIONS

This paper focuses on solving the multi- objective convex programming problems. It produces two methods to give the best compromise solution. First, it develops the existing Alia's method. Second, it integrates the first method with the distance based method for more flexibility. These methods depend on the new normal of objective functions. So, such problems are solved easily for any number of objectives. Generally, they are considered new directions toward future research in dealing with multi-objective convex programming problems. Next, their final solution can be used as an initial solution for the artificial intelligent algorithms in order to solve nonlinear multi-objective optimization problems.

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CONFLICT OF INTEREST

The authors declare no conflict of interest

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