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Numerical Solution of Fractional Integro-Differential Equation using Galerkin Method with Mamadu-Njoseh Polynomials

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ABSTRACT

This paper aims to employ the Mamadu-Njoseh Orthogonal Polynomial as a trial function through the Galerkin method to seek the approximate solution of fractional integro-differential equations. The Galerkin method is the weak finite element method formulation that effectively handles the residual equation. We have considered an approximate formulation in the Caputo sense in terms of the Mamadu-Njoseh polynomials. We have also presented our numerical evidences graphically showing the comparison of solutions between the analytic and approximate. The results revealed that the numerical procedure converges absolutely even at $N = 3$ to the exact solution. It was also observed that the use of Mamadu-Njoseh orthogonal Polynomial as a basis function via the Galerkin method solved the fractional integro-differential equation effectively and reliably as compared with other methods in literature. All computational frameworks are implemented by MAPLE 18 software.

Keywords: Mamadu-Njoseh polynomials, integro-differential equations, fractional derivatives, galerkin method, orthogonal polynomials

INTRODUCTION

The role of fractional derivatives in modern science and technology is quite fascinating. It has found applications in Biology, Physics, Economics and fluid mechanics(Podlubny, 1999; Bagley and Torvik, 2014). In the real sense, many physical phenomena are governed by fractional differential equations (FDEs), which have attracted the attention of many researchers (Dehestani et al., 2020; Ahmed and Salh, 2011; Arikoglu and Ozkol, 2009; Irandoust-Pakchin and Abdi-Mazrach, 2013; Oyedepo et al., 2016; Yang et al., 2014). However, many FDEs do not possess analytic solutions, giving room to numerical procedures (Sweillam et al., 2008; Sweillam and Khader, 2010). For example, the numerical scheme of the fractional reaction-diffusion model was considered by Akram et al. (2020). Many of the available numerical techniques for solving FDEs include the homotopy perturbation method(Saeed and Sdeq, 2010; Sweillam et al., 2008), Adomian decomposition method(Irandoust-Pakchin and Abdi-Mazrach, 2013), variational iteration method(Jafari and Daftardar-Gejji, 2006), collocation method (Khader, 2011; Rawashdeh, 2006; Yang et al., 2014) and homotopy analysis method (Mohammed, 2014).

The use of orthogonal polynomials in the approximation of a function in series expansion forms the basis of the approximation theory of differential equations(Fanaro, 1992). Moreover, several researchers have adopted orthogonal polynomials to solve most differential equations. For example, Khader (20110) employed the shifted Chebyshev polynomials for solving fractional diffusion-equation. In Canuto et al. (2006), the Laguerre polynomials were adopted via the spectral collocation method to solve nonlinear boundary value problems. On the other hand, the Mamadu-Njoseh polynomials define a complete set of orthogonal polynomials in the closed interval [-1,1]. The construction and properties of these polynomials can be seen in Njoseh and Mamadu(2016).

The Galerkin method is an effective and efficient numerical technique for solving linear and nonlinear problems. Many authors have adapted this method for different issues; Suk and Park(2019) used this method to find numerical solutions of transformed Richards equation in simulating flow problems. Similarly, Suk et al.(2020) applied the Galerkin FEM through the Dirichlet boundary conditions to solve the same flow problem.

For instance, Mamadu and Njoseh (2016) solved the Volterra integral equations with the orthogonal polynomials via the Galerkin method. On the other hand, Amaratunga (1994) proposed an augmented Galerkin method for the partial differential equation solution. The Central of the Galerkin method assumes a trial function $u(x)$ in terms of orthogonal polynomials such that it can be approximated using a linear combination suitable for the problem discussion.

In this article, the Galerkin method, with the aid of Mamadu-Njoseh polynomials, is considered for the solution of the linear fractional integrodifferential equation as given in equation (19).

PRELIMINARIES

We present in this section some vital definitions and essential preliminaries of fractional calculus relevant to this research.

Definition 2.1 (Riemann-Liouville Integral)
The second kind fractional integral of order α (otherwise called the left-sided Riemann-Liouville fractional integral) is defined as (Abramowitz and Stegun, 1964)

$$ {}_aJ_x^{\alpha} g = {}_aD_x^{-\alpha} g = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt, \ \mathbb{R}(\alpha) > 0. \qquad (1)$$

Similarly, the right-sided Riemann-Liouville fractional integral of order α is defined as

$$ {}_xJ_b^{\alpha} g = {}_xD_b^{-\alpha} g = \frac{1}{\Gamma(\alpha)} \int_x^b (x-t)^{\alpha-1} g(t)dt, \ \mathbb{R}(\alpha) > 0. \qquad (2)$$

Some Useful Properties [18 and 19]

1. If $g(t) = t^k$ (power function) and $a = 0$, then the left-sided Riemann-Liouville fractional integral becomes

$$ {}_0J_x^{\alpha} g = {}_0D_x^{-\alpha} g = \frac{\Gamma(k+1)x^{\alpha+k}}{\Gamma(k+\alpha+1)}, \ \mathbb{R}(\alpha) > 0, \mathbb{R}(k) + 1 > 0\,. $$

2. $ {}_aJ_x^{\alpha}\, {}_aJ_x^{\beta} g = {}_aJ_x^{\alpha+\beta} g = {}_aJ_x^{\beta}\, {}_aJ_x^{\alpha} g,$

$$ {}_xJ_b^{\alpha}\, {}_xJ_b^{\beta} g = {}_xJ_b^{\alpha+\beta} g = {}_xJ_b^{\beta}\, {}_xJ_b^{\alpha} g. $$

Definition 2.2. (Riemann-Liouville Derivative)
Suppose $[\alpha]$ is the integer domain of $\mathbb{R}(\alpha)$ and let $n = [\alpha] + 1$ such that $\mathbb{R}(n-1) +$ fractional part of $\mathbb{R}(\alpha) = 1$, where $\{\alpha\}$ will denote the fractional part of $\mathbb{R}(\alpha)$, that is, $\mathbb{R}(n-\alpha) = \mathbb{R}(1-\{\alpha\})$. Suppose $\mathbb{R}(\alpha) > 0$ then $n - \alpha = \{\alpha\} - \alpha + 1$. Thus, the left sided Riemann-Liouville fractional derivative of order α is defined as

$$ {}_aD_x^{\alpha} g(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} g(t)dt, & (n-1) < \alpha \leq n, \\ \left(\frac{d}{dx}\right)^{n-1} g(t) \text{ if } & \alpha + 1 = n \end{cases}, \qquad (3)$$

For example, if $g(t) = k$, a constant, then evaluating (3) will yield

$$ {}_aD_x^{\alpha} g(t) = \frac{k}{\Gamma(1-\alpha)(x-a)^{\alpha}}, \ \alpha \neq 1(2)\infty. $$

Definition 2.3. (Caputo Fractional Derivative)
If $\left(\frac{d}{dx}\right)^n$ in (3) is taken inside the integrand operating directing on $g(t)$, then the first kind or left-sided Caputo derivative is given as (Sweillam et al., (2008); Sweillam and Khader (2010)

$$ {}_{a}D_{x}^{\alpha}g(t)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-t)^{n-\alpha-1}g^{n}(t)dt, & (n-1)<\alpha\le n,\\ \left(\frac{d}{dx}\right)^{n-1}g(t) \text{if} & \alpha+1=n\end{cases} \quad (4)$$

where $g^n(t)$ denote the $n-th$ integer derivative of $g(t)$.

Useful Properties of the Caputo Fractional Derivative

Some properties of Caputo fractional derivative essential to this research are stated as follows(Saeed and sdeq, 2010; Sweillam et al., 2008; Swillam and Khader, 2010)

1. $D^{\alpha}K=0$, K is a constant (5)

2. $D^{\alpha}x^{m}=\begin{cases}0, & m\in\mathbb{N},\ m\ge[\alpha],\\ \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}x^{\beta-\alpha}, if & m\in\mathbb{N} \text{ and } m<[\alpha],\end{cases}$ (6)

where $[\alpha]\ge\alpha$ and $N=\{0,1,2,\ldots\}$.

3. $D^{\alpha}\left(\lambda g_1(x)+\mu g_2(x)\right)=\lambda D^{\alpha}g_1(x)+\mu D^{\alpha}g_2(x)$ (7)

3. MAMADU-NJOSEH POLYNOMIALS: DEFINITION AND PROPERTIES IN FRACTIONAL SENSE

The Mamadu-Njoseh polynomials $\left\{\varphi_n^{(\alpha)}(x)\right\}_{n=0}^{\infty}$, $\alpha\ge 0$, are orthogonal polynomials defined on the closed interval $[-1,1]$ and can be constructed using the following properties were based on these three properties (Njoseh and Mamdu, 2016; Mamadu an Njoseh, 2016)

$$\varphi_n^{(\alpha)}(x)=\sum_{r=0}^{n}C_r^{(n)}\frac{x^r}{\Gamma(r\alpha+1)}, \quad (8)$$

$$\left\langle\varphi_m^{(\alpha)}(x),\varphi_n^{(\alpha)}(x)\right\rangle=0, m\ne n, \quad (9)$$

$$\varphi_n^{(\alpha)}(x)=1+\alpha-\mathrm{x}. \quad (10)$$

These polynomials are orthogonal concerning the weight function $w(x)=\frac{x^r(1+x^2)}{\Gamma(r\alpha+1)}$. The orthogonality relation is given as

$$\frac{1}{\Gamma(r\alpha+1)}\int_{-1}^{1}x^r(1+x^2)\varphi_m^{(\alpha)}(x)\varphi_n^{(\alpha)}(x)dx=\binom{r+\alpha}{r}\delta_{mn}, \quad (11)$$

where δ_{mn} is the kronecter delta given as

$$\delta_{mn}=\begin{cases}0\\1\end{cases}.$$

The polynomials also satisfy

$$D^{a}\varphi_n^{(\alpha)}(x)=(-1)^{\mathrm{a}}\varphi_{n-a}^{(\alpha+a)}(x),\ \ \mathrm{a}=0(1)\mathrm{n}. \quad (12)$$

Let $u(x)\in L_w^2[-1,1]$ such that it is integrable on $[-1,1]$ with weight function $w(x)$, then it can be expressed in the series form

$$u(x)=\sum_{r=0}^{N}\mathrm{a_r}\varphi_r^{(\alpha)}(x), \quad (13)$$

where

$$\mathrm{a_r}=\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha+1+\alpha)}\int_{-1}^{1}x^r(1+x^2)\varphi_m^{(\alpha)}(x)\varphi_n^{(\alpha)}(x)dx,\ \ r=0,1,2,\ldots.$$

If we consider the first $(\mathrm{n}+1)$ Mamadu-Njoseh polynomials, we can write

$$u_N(x)\cong\sum_{r=0}^{N}\mathrm{a_r}\varphi_r^{(\alpha)}(x). \quad (14)$$

4. THE CONVERGENCE ANALYSIS OF THE COMPUTED FRACTIONAL DERIVATIVE OF $\varphi_n^{(\alpha)}(x)$.

The main objective of this section is to propose relevant theorems to derive a precise approximate formula of the Caputo fractional derivative of the Mamadu-Njoseh polynomials and study the convergence analysis and its truncation error.

Lemma 1.
Let $\varphi_n^{(\alpha)}(x)$ be given, then

$$D^\alpha \varphi_n^{(\alpha)}(x) = 0,\ \ \mathrm{n} = 0,1,2,\dots,([\alpha]-1), \alpha > 0. \tag{15}$$

Proof. It follows directly from implementing the properties of the Caputo fractional derivative (5) and (6).

Theorem 1.
Suppose $\alpha > 0$ and $u(x)$ being the analytic solution approximated by the Mamadu-Njoseh polynomials as (14) then, its Caputo fractional derivative has the form

$$D^\alpha(u_N(x)) \cong \sum_{\mathrm{i}=[\alpha]}^{\mathrm{N}} \sum_{\mathrm{m}=[\alpha]}^{i} \mathrm{a_m} \mathrm{w}_{\mathrm{i,m}}^{(\alpha)} \mathrm{x}^{\mathrm{m}-\alpha}, \tag{16}$$

where

$$\mathrm{w}_{\mathrm{i,m}}^{(\alpha)} = \frac{1}{\Gamma(m\alpha+1-\alpha)} \binom{i+\alpha}{i-m}.$$

Proof. Since the linear operation is valid for caputo fractional differentiation, we have that

$$D^\alpha(u_N(x)) = \sum_{r=0}^{N} \mathrm{a_r} D^\alpha \varphi_r^{(\alpha)}(x). \tag{17}$$

By lemma 1, we have that

$$D^\alpha \varphi_n^{(\alpha)}(x) = 0, \qquad \mathrm{n} = 0,1,2,\dots,([\alpha]-1), \alpha > 0.$$

Thus, for $\mathrm{n} = [\alpha],\dots \mathrm{N}$, by using (5) and (6) on (12), we have

$$D^\alpha \varphi_n^{(\alpha)}(x) = \sum_{\mathrm{m}=[\alpha]}^{i} \frac{(-1)^{\mathrm{n}}}{\Gamma(m\alpha+1-\alpha)} \binom{i+\alpha}{i-m} x^{m-\alpha}. \tag{18}$$

Using (17) and (18) leads to the required results.

5. NUMERICAL PROCEDURE USING THE GALERKIN METHOD

In this section, we present the Galerkin method to study the approximate solution of the Fredholm fractional integrodifferential equation (19) as given below:

$$D^\alpha\big(u(x)\big) = f(x) + \int_{-1}^{1} k(x,t)u(t)dt,\ \ x+1 \geq 0, t-1 \leq 0, \tag{19}$$

with prescribed initial conditions

$$u^n(0) = A_n,\ \ (n-1) < \alpha \leq n,\ n \in \mathbb{N}, \tag{20}$$

where $D^\alpha\big(u(x)\big)$ is in Caputo sense, $f(x)$ is the source term, $u(x)$ is the unknown function, x and t are variables defined in the closed interval $[-1,1]$ and $k(x,t)$, being the kernel.

The implementation of the Galerkin procedure is aided by the following steps:

1. Substitute (13) into (19) to obtain

$$D^\alpha\Big(\sum_{r=0}^{N} \mathrm{a_r}\varphi_r^{(\alpha)}(x)\Big) - \int_{-1}^{1} k(x,t)\Big(\sum_{r=0}^{N} \mathrm{a_r}\varphi_r^{(\alpha)}(t)\Big)dt = f(x). \tag{21}$$

2. Multiply both sides of (21) by $\varphi_j^{(\alpha)}(x)$, $\mathrm{j} = 0,1,2,\dots,\mathrm{n}$, and integrate within the interval [a, b] with respect to x, that is,

$$\int_a^b \left(D^\alpha\left(\sum_{r=0}^{N} \mathrm{a_r}\varphi_r^{(\alpha)}(x)\right) - \int_{-1}^{1} k(x,t)\left(\sum_{r=0}^{N} \mathrm{a_r}\varphi_r^{(\alpha)}(t)\right)dt\right)\varphi_j^{(\alpha)}(x)\mathrm{dx} = \int_a^b \mathrm{f(x)}\varphi_j^{(\alpha)}(x)\mathrm{dx} \qquad (22)$$

3. Write (22) in the matrix form

$$A\bar{x} = b, \qquad (23)$$

where

$A = a_{ij} = \int_a^b \left(D^\alpha\left(\sum_{r=0}^{N} \mathrm{a_r}\varphi_r^{(\alpha)}(x)\right) - \int_{-1}^{1} k(x,t)\left(\sum_{r=0}^{N} \mathrm{a_r}\varphi_r^{(\alpha)}(t)\right)dt\right)\varphi_j^{(\alpha)}(x)\mathrm{dx},\ ,\ \mathrm{j} = 0(1)\mathrm{n},$

$\bar{x} = x_i = (x_1, x_2, x_3, \dots, x_n)^T,$

$b = b_j = \int_a^b \mathrm{f(x)}\varphi_j^{(\alpha)}(x)\mathrm{dx},\ \mathrm{j} = 0(1)\mathrm{n}.$

4. Solve the above system (23) via the Gaussian elimination method to obtain the values of $a_i, i = 0(1)n$.
5. Substitute the values of the $a_i, i = 0(1)n$, into (13) to obtain the approximate solution.

6. NUMERICAL APPLICATIONS

This section illustrates the accuracy and reliability of the method discussed. To this end, we shall use two examples from literature. These examples are solved by applying our laid down methodology.

Example 1. Consider the following linear fractional integro-differential equation

$$D^{5/3}u(x) = \frac{3\sqrt{3}\Gamma(2/3)}{\pi} - \frac{1}{5}x^2 - \frac{1}{4}x + \int_0^1 (xt + x^2t^2)u(t)dt,\ x \geq 0, t \leq 1, \qquad (24)$$

with initial conditions $u(0) = u'(0) = 0$. The analytic solution is $u(x) = x^2$.
Applying the Galerkin method with the aid of Mamadu-Njoseh polynomials of $\varphi_n^{(\alpha)}(x)$, $\mathrm{n} = 0,1,2, \dots \mathrm{N}$, at $N = 3$ to the problem (24), we obtain four (4) linear systems of equations with four (4) unknowns $a_i, i = 0(1)3$. Solving the system via the Gaussian elimination method we obtain the values of $a_i, i = 0(1)3$, as:

$$a_0 = \frac{13}{5}, a_1 = -4, a_2 = \frac{12}{5}, a_3 = 0$$

Substituting the above values into (13), we obtain the approximate solution $u(x) = x^2$, which is same as the analytic solution.

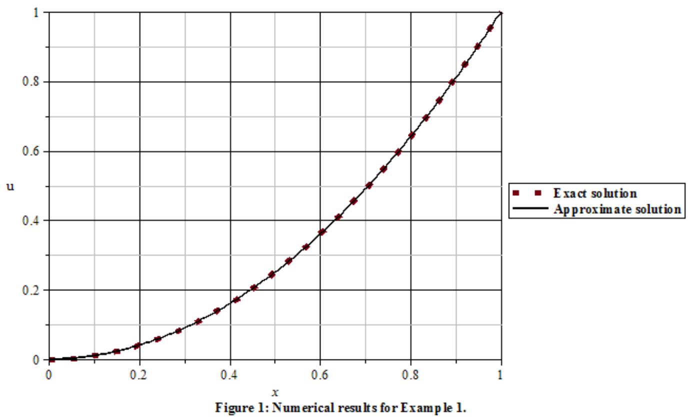

Figure 1: Numerical results for Example 1.

Example 2. Consider the following linear fractional integro-differential equation

$$D^{1/2}u(x) = \frac{(3/8)x^{3/2} - 2x^{1/2}}{\sqrt{\pi}} + \int_0^1 xtu(t)dt,\ x \geq 0, t \leq, \qquad (25)$$

with initial $u(0) = 0$. The analytic solution is given as $u(x) = x^2 - x$.

Again, applying the Galerkin method with the aid of Mamadu-Njoseh polynomials of $\varphi_n^{(\alpha)}(x)$, $\mathrm{n} = 0,1,2, \dots \mathrm{N}$, at $N = 3$ to problem (25), the values of $a_i, i = 0(1)3$, are given as:

$$a_0 = \frac{18}{5}, a_1 = -6, a_2 = \frac{12}{5}, a_3 = 0$$

Substituting the above values into (13), we obtain the approximate solution $u(x) = x^2 - x$, which is same as the analytic solution.

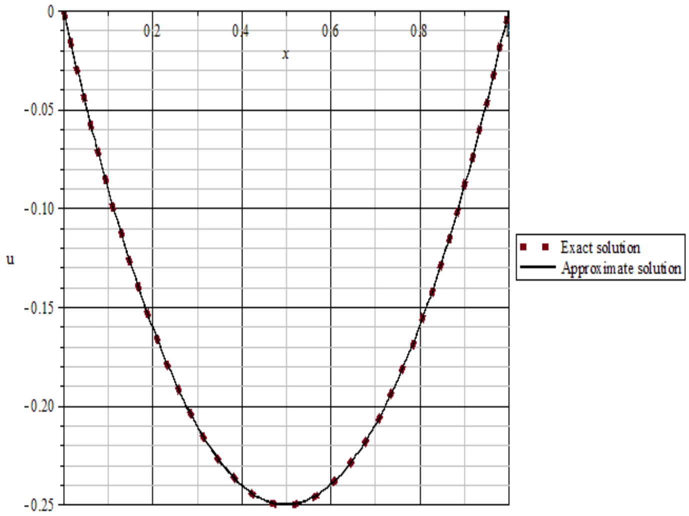

Figure 2: Numerical results for Example 2.

7. DISCUSSION OF RESULTS

The resulting numerical evidence using the laid down methodology shows that the method converges absolutely to the analytic solution. Results are compared with the exact and approximate solutions to show the method's accuracy, validity, and reliability, as shown in Figures 1 and 2. Also, the numerical evidence is in perfect arrangement with those found in the literature [17].

8. CONCLUSION

This article has implemented an accurate, valid and reliable numerical procedure for the linear fractional Fredholm integro-differential equation. We have also considered an approximate formulation in the Caputo sense of the Mamadu-Njoseh polynomials. The results reveal that the procedure converges absolutely even at $N = 3$. We express the solution as a truncated Mamadu-Njoseh series so that any mathematical software can easily solve it without any computational stress. From the examples considered above, it is evident that the method is very accurate as it produces the analytic solution. We have also presented our numerical evidence graphically, comparing solutions between the analytic and approximate. All computational frameworks are implemented by MAPLE 8 software.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

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